# Convex Choice*

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**Abstract**

For multidimensional Euclidean type spaces, we study convex choice: from any choice set, the set of types that make the same choice is convex. We establish that, in a suitable sense, this property characterizes the sufficiency of local incentive constraints. Convex choice is also of interest more broadly, e.g., in cheap-talk games. We tie convex choice to a notion of directional single-crossing differences (DSCD). For an expected-utility agent choosing among lotteries, DSCD implies that preferences are either one-dimensional or must take the affine form that has been tractable in multidimensional mechanism design.

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# 1. Introduction

A fundamental property in economic models with one-dimensional private information is the Spence-Mirrlees single-crossing property. Roughly equivalent to "interval choice"—the set of types for which an action is optimal is an interval—the single-crossing property has many uses. For example, it reduces incentive compatibility (IC) to local IC. Local IC requires only that "nearby" types have no incentive to mimic each other. Sufficiency of local IC is central to the tractability, and hence success, of influential paradigms like signaling and mechanism design.

Our paper is concerned with multidimensional environments. Consider an agent with utility function $u(a, \theta)$, where $a \in A$ is an allocation or choice variable and $\theta \in \Theta$ is the agent's type or preference parameter. Let $\Theta \subset \mathbb{R}^n$ be convex. We study a property we call *convex choice*: for any choice set $B \subset A$ and any allocation $a \in B$, the set of types for which $a$ is (uniquely) optimal is convex.[1]

We view convex choice as compelling. There is a sense in which it characterizes the sufficiency of local IC. More precisely, absent indifferences, convex choice characterizes when IC between any two types is equivalent to local IC along the line segment between those two types (Proposition 1). The necessity of convex choice for such "integration up" of local IC suggests that absent convex choice, a mechanism design model is unlikely to be tractable. Convex choice is relevant for other reasons as well. For instance, it delivers an extension of Saks and Yu's (2005) result on the implementablity of allocation rules in settings with transfers (Proposition 2). It also offers a natural generalization of "interval equilibria" in multidimensional versions of Crawford and Sobel's (1982) famous cheap-talk model.

Proposition 3 shows that convex choice is essentially equivalent to a form of single crossing that we term *directional single-crossing differences* (DSCD). Modulo details, DSCD requires that for any two (undominated) allocations, the set of types that prefer one to the other is defined by a half-space. Importantly, the orientation of the half-space can vary with the allocation pair. When $A \subset \mathbb{R}^n$, leading examples of DSCD are the quadratic-loss utility $u(a, \theta) = -\|a - \theta\|^2$, where $\|\cdot\|$ is the Euclidean norm, and the constant-elasticity-of-substitution utility $u(a, \theta) = \left(\sum_{i=1}^{d} (a_i)^r \theta_i\right)^s$ with parameters $r \in \mathbb{R}$ and $s > 0$.

Studies of multidimensional mechanism design frequently assume the *affine form* $u(a, \theta) = v(a) \cdot \theta + w(a)$ (e.g., Armstrong, 1996; Rochet and Choné, 1998; Manelli and Vincent, 2007;

[1] Our notion relates the preferences of different types. It is distinct from "convex preferences" in decision theory, which refers to the following property of a single preference relation: if $a$ is preferred to $b$, then convex combinations of $a$ and $b$ (in some suitable sense) are also preferred to $b$.

Kleiner, 2022). Our approach delivers a novel perspective on this specification. Propositions 4 and 5 show that if the convex choice/DSCD requirement is strengthened to hold for expected utility over allocation lotteries, then—subject to some regularity conditions—either the setting is effectively one-dimensional, or preferences can be represented in the affine form. A provocative suggestion, then, is that when allocation lotteries are important, multidimensional mechanism design is unlikely to be generally tractable beyond the affine form. The consideration of lotteries may be inescapable when stochastic mechanisms are allowed or there are other agents whose private information also influences a deterministic mechanism.

**Related Literature.** Convex choice is closely related to a notion of Grandmont (1978), as elaborated below. Our Proposition 3 is thus also similar to his characterization Proposition (p. 322). Beyond that, our work is distinct; in particular, his interest was in social choice, he did not relate convex choice to the sufficiency of local IC or to applications like cheap talk, and he had no analog to our characterization of DSCD over lotteries.

Our work also relates to Carroll (2012), Kartik, Lee, and Rappoport (2024b), and McAfee and McMillan (1988). We detail these connections subsequently, as well as connections to some papers on implementability in mechanism design and on cheap talk.

## 2. Convex Choice and Applications

There is an agent with type $\theta \in \Theta \subset \mathbb{R}^n$, where $\Theta$ is convex.[2] We write $\theta_{(i)}$ for the $i$-th coordinate of $\theta$, and $\theta > \theta'$ if $\theta_{(i)} \geq \theta'_{(i)}$ for all $i$ with strict inequality for some $i$. We denote the *line segment* between types $\theta'$ and $\theta''$ by

$$\ell(\theta', \theta'') := \{\theta : \exists \lambda \in [0,1] \text{ with } \theta = \lambda\theta' + (1-\lambda)\theta''\}.$$

The agent must take an action, or choose an allocation, $a \in A$. The agent's preferences are given by the utility function $u : A \times \Theta \to \mathbb{R}$.

**Definition 1.** $u$ has *convex choice* if for all $B \subset A$ and $a \in B$,

$$\left\{\theta \in \Theta : \{a\} = \arg\max_{b \in B} u(b, \theta)\right\} \text{ is convex.}$$

In other words, convex choice requires that from any choice set, the set of types that find an action uniquely optimal is convex. It is equivalent to only consider all binary choice sets,

[2] In this paper, the "$\subset$" symbol means "weak subset".

as the intersection of convex sets is convex. Convex choice implies that the preferences of any type $\theta \in \ell(\theta', \theta'')$ are "between" those of $\theta'$ and $\theta''$, in the sense of Grandmont (1978).[3]

Convex choice is related to but different from Kartik et al.'s (2024b) "interval choice". They require that if $\theta'' > \theta' > \theta$ and some action is optimal for $\theta$ and $\theta''$, then it is also optimal for $\theta'$. (Those authors use weak optimality whereas we use strict optimality; this difference is minor.) Absent indifferences, convex and interval choice are equivalent when $\Theta \subset \mathbb{R}$, but they are incomparable when $\Theta$ is multidimensional.[4] Convex sets are, of course, an (alternative) salient generalization of one-dimensional intervals to multiple dimensions. We will discuss yet another alternative, connected sets, subsequently.

To define our notions of incentive compatibility, let $N_\theta \subset \Theta$ denote an open neighborhood—simply neighborhood hereafter—of type $\theta$ in the relative topology of $\Theta$. Throughout this paper, we focus on direct mechanisms $\Theta \to A$. Stochastic mechanisms are subsumed by taking $A$ to be a lottery space.

**Definition 2.** A mechanism $m : \Theta \to A$ is

1. *incentive compatible* (IC) if $u(m(\theta), \theta) \geq u(m(\theta'), \theta)$ for all $\theta, \theta' \in \Theta$;
2. *locally IC* if for each $\theta \in \Theta$ there is a neighborhood $N_\theta \subset \Theta$ such that

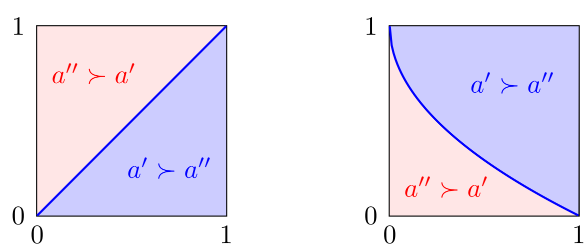
$$\forall \theta' \in N_\theta : \; u(m(\theta), \theta) \geq u(m(\theta'), \theta) \text{ and } u(m(\theta'), \theta') \geq u(m(\theta), \theta'). \tag{LIC}$$

We will also refer to (local) IC of mechanisms defined on a subset of the type space $\Theta' \subset \Theta$; that simply refers to Definition 2 with $\Theta'$ in place of $\Theta$.

IC is a standard and fundamental property. Our formulation of local IC follows Carroll

[3] Grandmont's betweenness notion is stated for binary relations that need not be transitive; so, in his setting, choice from non-binary choice sets may not be well-defined. His notion also imposes some requirements concerning indifference. Ignoring indifferences and assuming transitivity, convex choice is equivalent to the preferences of any $\theta \in \ell(\theta', \theta'')$ being between those of $\theta'$ and $\theta''$.

[4] Ignoring indifferences, convex choice is equivalent to "interval choice on every line segment", which is neither stronger nor weaker than interval choice on the full type space when $\Theta$ is multidimensional. Consider the following the two examples with $\Theta = [0, 1]^2$ and $A = \{a', a''\}$:




On the left, there is convex but not interval choice; on the right, there is interval but not convex choice.

(2012): for every type $\theta$, there is a set of nearby types for which (i) $\theta$ cannot profitably mimic those types and (ii) those types cannot profitably mimic $\theta$. Requirement (ii) here owes to the non-discrete type space, and cannot be dispensed with.[5] It is less demanding than imposing only requirement (i) but requiring that all neighborhoods $N_\theta$ in (LIC) have "size" bounded away from zero.[6]

Plainly, IC implies local IC (because $\Theta$ is a neighborhood of every type). The converse is not true:

**Example 1.** $\Theta = [0,1]$, $A = \{a', a''\}$, and $u(a,\theta) = \mathbb{1}\{a = a'', \theta \in (1/3, 2/3)\}$. Note that convex choice fails. The mechanism $m(\theta) = a'$ if $\theta \leq 1/3$ and $m(\theta) = a''$ if $\theta > 1/3$ is not IC. But it is locally IC: for every $\theta \neq 1/3$, there is a neighborhood of $\theta$ on which $m(\cdot)$ is constant; whereas for type 1/3, neither does it prefer to mimic any type nor does any nearby type prefer to mimic it.

We will see that, in a suitable sense, convex choice is necessary for local IC to imply IC. While it is also sufficient when preferences are strict, the next example shows that indifferences are a threat.

**Example 2.** $\Theta = [0,1]$, $A = \{a', a''\}$, and $u(a,\theta) = \mathbb{1}\{a = a'', \theta = 1\}$. Despite convex choice, the mechanism $m(\theta) = a''$ if and only if $\theta \leq 1/2$ is not IC. Yet it is locally IC: only type $\theta = 1$ can profitably mimic any other type, but $m(\cdot)$ is constant on $(1/2, 1]$.

The problem in Example 2 is "thick" indifferences. The next definition allows us to pinpoint the issue.

**Definition 3.** $u$ has *regular indifferences* if for all $a', a'' \in A$ and $\theta', \theta'' \in \Theta$, $u(a', \theta') > u(a'', \theta')$ and $u(a', \theta'') = u(a'', \theta'')$ imply there is a sequence $\{\theta_n\} \to \theta''$ such that $\theta_n \in \ell(\theta', \theta'')$ and $u(a', \theta_n) > u(a'', \theta_n)$ for all $n$.

An implication of regular indifferences is that for any two actions, if some type strictly prefers one of the actions then the set of types that are indifferent has empty interior. We view regular indifferences as a weak requirement; in particular, it trivially holds when

[5] Here is an example. $\Theta = [0,1]$, $A = \{a', a''\}$, $u(a', \cdot) < u(a'', \cdot)$, and $m(\theta) = a'$ if and only if $\theta < 1/2$. For every type $\theta < 1/2$, there is $\varepsilon > 0$ such that all types in $[0, \theta + \varepsilon)$ receive the same allocation $a'$. Hence, for every $\theta$, there is a neighborhood in which $\theta$ has no incentive to mimic any type in that neighborhood. Local IC fails, however, because any neighborhood of type 1/2 contains some types strictly below, all of which would profitably mimic 1/2. If we did not rule this out in local IC, then the example would contradict Proposition 1. Carroll (2012, p. 669) makes an analogous point in his framework.

[6] That is, there is $\varepsilon > 0$ such that for all $\theta \in \Theta$, the $\varepsilon$-ball $\{\theta' \in \Theta : \|\theta - \theta'\| < \varepsilon\}$ is contained in $N_\theta$, where $\|\cdot\|$ is the Euclidean norm.

there are no indifferences. Regular indifferences is also satisfied by, for example, $A \subset \mathbb{R}^n$ and $u(a,\theta) = \left(\sum_{i=1}^{n}(a_{(i)})^r \theta_{(i)}\right)^s$ with parameters $r \in \mathbb{R}$ and $s > 0$. On the other hand, Example 2 is a violation.

**Proposition 1.** *The following are equivalent:*

1. *$u$ has convex choice and regular indifferences;*
2. *for any line segment $\ell \subset \Theta$ and any mechanism $m : \ell \to A$, if $m$ is locally IC then it is IC.*

(All proofs are in the appendices.)

The proposition says that convex choice and regular indifferences jointly characterize when IC between any two types $\theta$ and $\theta'$ can be determined by just checking local IC along the line segment $\ell(\theta, \theta')$. Such "integration up" is a common strategy used to verify IC.

Local IC on the full type space implies local IC on every line segment.[7] Hence, a corollary of Proposition 1 is that convex choice and regular indifferences imply that on the full type space, local IC is sufficient for IC. But sufficiency of local IC on the full type space does not require convex choice, even assuming regular (or even no) indifferences. We show in Propositions 7 and 8 of Appendix A.7 that modulo some details (such as the nature of indifferences), sufficiency of local IC on the full type space is characterized by "*connected* choice". Connected choice demands that any action from any choice set is optimal for a connected set of types; this is evidently weaker than convex choice.

Notwithstanding, we view the sufficiency of local IC on the full type space as generally insufficient to render multidimensional incentive constraints tractable. In particular, it seems intractable to verify IC between two types by checking local IC along (the typically uncountable set of) all paths between those types. Instead, tractability does obtain if IC can be verified via local IC along their (unique) line segment; that is a one-dimensional task. Hence our interest in the more demanding form of sufficiency of local IC embodied in Proposition 1.[8]

Figure 1 illustrates the difference between sufficiency of local IC on the full type space and on all line segments. The figure's example violates convex choice, but because there is

[7] Formally, if $m : \Theta \to A$ is locally IC, then for every line segment $\ell$, the mechanism $m' : \ell \to A$ defined by $m'(\theta) := m(\theta)$ is locally IC.

[8] Another perspective may also be helpful. Consider an environment with a one-dimensional type space, $\Theta'$, in which local IC does not imply IC. Let us stipulate that this environment is not tractable. It would be perverse to then say that embedding it into a multidimensional environment with type space $\Theta \supset \Theta'$ could result in the multidimensional environment being tractable if, on $\Theta$, local IC implies IC. Figure 1 below gives an example of such an embedding.

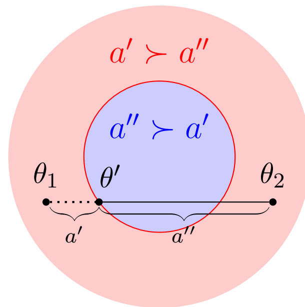



**Figure 1:** $\Theta$ is the large disc and $A = \{a', a''\}$. Blue (inner disc) types prefer $a''$ to $a'$; red (outer region) types prefer $a'$ to $a''$. On the full type space, local IC implies IC. However, a mechanism that allocates $a'$ on the dotted line segment (including to $\theta'$) and $a''$ on the solid one is locally IC, but not IC, on the line segment from $\theta_1$ to $\theta_2$.

connected choice, on the full type space any locally IC mechanism is IC.[9] However, as shown in the figure, there are mechanisms in which local IC holds along the line segment $\ell(\theta_1, \theta_2)$ even though $\theta_2$ would mimic $\theta_1$.[10]

Examples 1 and 2 are illustrative of the general argument for why both convex choice and regular indifferences are necessary for Proposition 1's second statement. Here is a heuristic sketch for why convex choice is sufficient when there are no indifferences. Take any $\theta$ and $\theta'$. Consider a fine grid of types $\theta = \theta_1, \ldots, \theta_m = \theta'$ traversing the line segment $\ell(\theta, \theta')$. We can regard local IC as implying $u(m(\theta), \theta) \geq u(m(\theta_2), \theta)$, $u(m(\theta_2), \theta_2) \geq u(m(\theta_3), \theta_2)$, and $u(m(\theta_3), \theta_3) \geq u(m(\theta_2), \theta_3)$. Absent indifferences, all those inequalities are strict. Convex choice then implies $u(m(\theta), \theta) > u(m(\theta_3), \theta)$; otherwise, both $\theta$ and $\theta_3$ strictly (absent indifferences) prefer $m(\theta_3)$ to $m(\theta_2)$, hence so must $\theta_2$, a contradiction. Iterating this logic, using the combination of local IC and convex choice each time, yields $u(m(\theta), \theta) > u(m(\theta_i), \theta)$ for all $i = 2, \ldots, m$. Consequently, $u(m(\theta), \theta) > u(m(\theta'), \theta)$.

Let us now compare our approach with that of Carroll (2012) in his "cardinal type space" analysis.[11] We view the approaches as complementary: Carroll equates types with

[9] Formally, this follows from Proposition 7 in Appendix A.7. The logic in the current example is as follows: take any non-IC mechanism in which type $\theta_2$ prefers to mimic $\theta_1$. As the set of types that prefer each action in $A = \{a', a''\}$ is path-connected, there is a path from $\theta_2$ to $\theta_1$ along which there is at most one point at which the preference between the two actions flips. Since there must also be some (possibly distinct) point on the path at which the allocation flips, local IC is violated at this point.

[10] In Figure 1's example, one can also show that

$$\text{if } m : \Theta \to A \text{ is locally IC on every line segment, then it is IC (on } \Theta). \tag{1}$$

Since the figure does not have convex choice, it follows that the second statement of Proposition 1 is more demanding than (1); note the different order of quantifiers.

[11] He also considers "ordinal type spaces", which are less comparable to our framework.

preferences (more precisely, utilities) and studies properties of the type space that guarantee sufficiency of local IC; instead, we fix an abstract (convex Euclidean) type space and study properties of the utility function. Berger, Müller, and Naeemi (2017, p. 3) provide an insightful discussion of "parameter representations", like ours, versus "domain representations", like Carroll's. We add that the flexibility in choosing a parameter representation (the type space and the utility function) has advantages and disadvantages, roughly corresponding to whether one deploys Proposition 1 for sufficiency or necessity of convex choice. Since we are more interested by sufficiency, we are inclined to view it as an advantage, as illustrated below.

Mathematically, Proposition 1 subsumes Carroll's (2012) Proposition 1. To see why, recall that Carroll assumes each $a \in A \subset \mathbb{R}^n$ is a probability vector over a finite set of $n$ outcomes, and he associates types with their utility vectors over those outcomes. We subsume his framework by setting $u(a, \theta) = a \cdot \theta$. Our notion of local IC (and IC) is then equivalent to his. Carroll's Proposition 1 says that given convexity of $\Theta$, local IC implies IC. The conclusion follows from our Proposition 1 because $u$ in this specification has convex choice and regular indifferences. Our result is more general because we do not require finite outcomes, and because utility functions that are nonlinear in types (and satisfy the conditions for Proposition 1) need not result in convex type spaces in Carroll's (2012) representation. Moreover, even in cases where both results are applicable, it can be easier to verify that a parameter representation has convex choice. Consider, for example, $\Theta = [0, 1]$, a finite set of actions $A \subset \mathbb{R}$, and $u(a, \theta) = -(a - \theta)^2$. It is elementary that convex choice is satisfied. By contrast, checking that the domain representation à la Carroll (2012) has a convex type space is more involved—albeit, hardly insurmountable in this simple example.

So far we have emphasized the sufficiency of local IC. But convex choice is relevant and useful more broadly. We next discuss two other applications.

**Convex Choice and Implementability.** In mechanism design with transfers, an important question is whether a given allocation rule is implementable. Formally, let $A \equiv Y \times \mathbb{R}$, where $Y$ is a "physical" allocation space and $\mathbb{R}$ is the space of transfers. Writing $a \equiv (y, t)$, assume $u((y, t), \theta) = \tilde{u}(y, \theta) - t$ for some function $\tilde{u}$. An *allocation rule* $\upsilon : \Theta \to Y$ is *implementable* if there exists a transfer rule $\tau : \Theta \to \mathbb{R}$ such that $m \equiv (\upsilon, \tau)$ is IC. We restrict attention to allocation rules with finite range. Rochet (1987) establishes that an allocation rule is implementable if and only if it is "cyclically monotone". A less demanding condition, which only considers pairs of types, is that of *weak monotonicity*:

$$\forall \theta, \theta' \in \Theta : \ \tilde{u}(\upsilon(\theta), \theta) - \tilde{u}(\upsilon(\theta'), \theta) \geq \tilde{u}(\upsilon(\theta), \theta') - \tilde{u}(\upsilon(\theta'), \theta')$$

It is straightforward that weak monotonicity is necessary for implementability. For $Y \subset \mathbb{R}^n$ with dot product valuations, $\tilde{u}(y, \theta) = y \cdot \theta$, Saks and Yu (2005) establish that if $\Theta$ is convex, then weak monotonicity is also sufficient for implementability; see Bikhchandani, Chatterji, Lavi, Mu'alem, Nisan, and Sen (2006) and Archer and Kleinberg (2014) as well. Using a result of Berger et al. (2017, Theorem 5), the notion of convex choice yields the following generalization (which does not assume $Y \subset \mathbb{R}^n$):

**Proposition 2.** *Assume $u((y,t),\theta) = \tilde{u}(y,\theta) - t$ has convex choice, regular indifferences, and is continuous in $\theta$. If an allocation rule with finite range is weakly monotone, then it is implementable.*

**Convex Choice in Cheap Talk.** In cheap-talk or costly-signaling applications, it is natural to ask whether, in equilibrium, the set of types that choose a particular signal is convex. Modulo details about indifferences, the key condition to guarantee that all equilibria have this "convex partitional" property is that the sender's utility have convex choice.

Indeed, there has been interest in extending Crawford and Sobel's (1982) classic result on interval equilibria in one-dimensional cheap talk to multiple dimensions. In a working paper version of Levy and Razin (2007), Levy and Razin (2004) assume $\Theta = A = \mathbb{R}^2$ and sender utility $u(a,\theta) = \left(\sum_{i \in \{1,2\}} \alpha_{(i)} \left|a_{(i)} - \left(b_{(i)} + \theta_{(i)}\right)\right|^p\right)^{\frac{1}{p}}$ with parameters $\alpha \in \mathbb{R}_{++}$, $b \in \mathbb{R}^2$, and $p > 1$. It can be verified that this specification has convex choice if and only if $p = 2$ (cf. Remark 1 in the next section). Consistent with that, Levy and Razin (2004) do not show that all equilibria are convex partitional; rather, in their Section 4.1, they identify parameters under which they can construct some such equilibria.[12] Similarly, the class of sender utilities in Chakraborty and Harbaugh (2007) do not assure convex choice, and they only show existence of "comparative cheap-talk" equilibria that are convex partitional.

Also germane are a few papers on common-interest cheap-talk games with an infinite type space but a finite message space. Jäger, Metzger, and Riedel (2011) posit $u(a,\theta) = -l\left(\|a - \theta\|\right)$ for an arbitrary norm $\|\cdot\|$ and increasing function $l(\cdot)$, but in their Corollary 1 on optimal equilibria being convex partitional (modulo indifferences—which we ignore in the rest of this paragraph), they assume the Euclidean norm, which guarantees convex choice. They note that other norms can be problematic; we will see that weighted Euclidean norms would also work, but none others (Remark 1).[13] Saint-Paul (2017, Theorem 2) establishes

[12] In other working paper versions, the authors assume $p = 2$ and show that in this case all equilibria are convex partitional.

[13] Strictly speaking, Jäger et al.'s (2011) Example 1/Figure 1 with the maximum norm does not violate convex choice—rather, the issue there is about indifferences—but perturbations of their example's actions would illustrate a failure of convex choice.

that optimal equilibria are convex partitional under a weaker condition than Jäger et al. (2011). Saint-Paul's condition implies the directional single-crossing property discussed in the next section, which we show essentially characterizes convex choice (Proposition 3). Lastly, Sobel (2016, Proposition 3) extends Saint-Paul (2017, Theorem 2) by showing that convex choice guarantees that equilibria are convex partitional.

# 3. Directional Single Crossing

To unpack which preferences have convex choice, we next develop a connection with single crossing.

**Definition 4.** A function $f : \Theta \to \mathbb{R}$

1. is *directionally single crossing (DSC)* if there is $\alpha \in \mathbb{R}^n \setminus \{0\}$ such that for all $\theta, \theta' \in \Theta$,

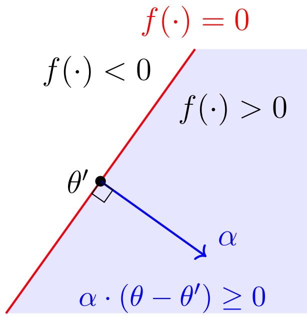
$$(\theta - \theta') \cdot \alpha \geq 0 \implies \text{sign}\,(f(\theta)) \geq \text{sign}\,(f(\theta'))\,;$$

2. *strictly violates DSC* if for all $\alpha \in \mathbb{R}^n \setminus \{0\}$, there are $\theta, \theta', \theta'' \in \Theta$ with $\alpha \cdot \theta < \alpha \cdot \theta' < \alpha \cdot \theta''$ such that

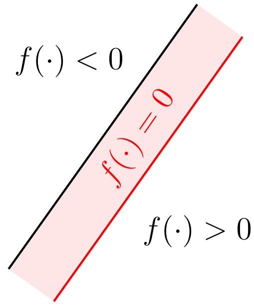
$$\min\{f(\theta), f(\theta'')\} > 0 > f(\theta') \quad \text{or} \quad \max\{f(\theta), f(\theta'')\} < 0 < f(\theta').$$

DSC says that for any $\theta$ and $\theta'$ such that $\theta$ is to the right of the hyperplane passing through $\theta'$ in the direction of $\alpha$ (i.e., with normal vector $\alpha$), we have $\text{sign}\,(f(\theta)) \geq \text{sign}\,(f(\theta'))$. Hence, we will sometimes say more explicitly that a function is DSC in the direction $\alpha$. DSC is equivalent to the sets $\{\theta : f(\theta) < 0\}$ and $\{\theta : f(\theta) > 0\}$ being either empty, full (i.e., all of $\Theta$), or intersections of $\Theta$ with half-spaces defined by parallel (possibly identical) hyperplanes. See Figure 2: the left panel depicts a typical DSC function with a "thin" zero set, illustrating the geometry of the definition; the middle panel depicts a DSC function with a "thick" zero set; and the right panel depicts a violation of DSC. Note that DSC implies that along any line segment, the sign of the function is monotonic.

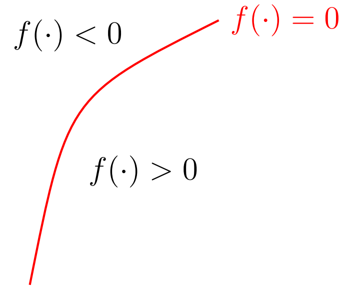
Regarding part 2 of Definition 4, note that a strict violation of DSC is slightly stronger than just violating DSC. To illustrate, consider $\Theta = [-1, 1]$ and $f(\theta) = |\theta|$. This function is not DSC because its sign is not monotonic. But it does not strictly violate DSC. If, instead, $f(0) < 0$ (maintaining $f(\theta) > 0$ for $\theta \neq 0$), then it does strictly violate DSC. The violation of DSC in Figure 2c is also a strict violation.



(a) DSC



(b) DSC with thick zero set



(c) Not DSC

**Figure 2:** Directional single crossing with $\Theta \subset \mathbb{R}^2$.

**Definition 5.** The utility function $u$

1. has *directionally single-crossing differences (DSCD)* if for all $a, a' \in A$, the difference $D_{a,a'}(\theta) := u(a, \theta) - u(a', \theta)$ is DSC;
2. *strictly violates DSCD* if there are $a, a' \in A$ such that $D_{a,a'}(\theta)$ strictly violates DSC.

DSCD can be viewed as saying that for any pair of actions $a$ and $a'$, the strict-preference sets $\{\theta : u(a, \theta) > u(a', \theta)\}$ and $\{\theta : u(a, \theta) < u(a', \theta)\}$ are parallel half-spaces, each either open or closed, intersected with the type space. Importantly, the direction of the hyperplanes defining these half-spaces can vary across action pairs.

Here are two leading families of DSCD when $A \subset \mathbb{R}^n$: (i) weighted Euclidean preferences, where $u(a, \theta)$ is any decreasing function of $(a - \theta)W(a - \theta)^T$ with $W$ a $n \times n$ symmetric positive definite matrix; and (ii) constant-elasticity-of-substitution (CES) preferences, where $A, \Theta \subset \mathbb{R}^n_+$ and $u(a, \theta) = \left(\sum_{i=1}^n (a_{(i)})^r \theta_{(i)}\right)^s$ with parameters $r \in \mathbb{R}$ and $s > 0$.[14] In either case, adding a type-independent function $w(a)$ preserves DSCD.[15] With that addition, the CES family subsumes $u(a, \theta) = a \cdot \theta + w(a)$, which is frequently used in multidimensional mechanism design, and which we will return to.

*Remark* 1. Consider norm-based preferences, where $A \subset \mathbb{R}^n$ and $u(a, \theta) = -l(\|a - \theta\|)$ for a strictly increasing function $l(\cdot)$ and some norm $\|\cdot\|$ on $\mathbb{R}^n$. If the norm is weighted Euclidean (i.e., there is a symmetric positive definite $n \times n$ matrix $W$ such that $\|x\| = \sqrt{xWx^T}$ for all $x \in \mathbb{R}^n$), then we have weighted Euclidean preferences and hence DSCD. Conversely, if

[14] The superscript $^T$ denotes transposition. Family (i) satisfies DSCD with the direction $\alpha = W(a - a')^T$ and family (ii) with $\alpha = a^r - (a')^r$.

[15] More generally, DSCD may not be preserved by such addition. But the current utility families don't just have DSCD, they have directionally monotonic differences: for any pair of actions, the utility difference is monotonic—rather than just single crossing—in the relevant direction.

the norm is not weighted Euclidean, then DSCD is violated if $A \cap \Theta$ has nonempty interior; Appendix A.3 provides a proof.

The following result shows that DSCD "almost" characterizes convex choice.

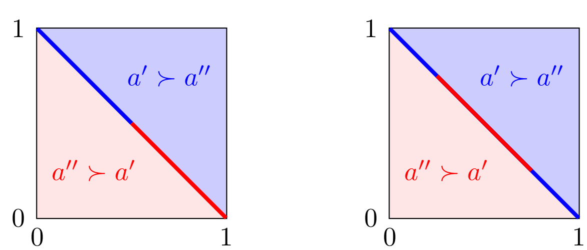
**Proposition 3.** *If $u$ has DSCD, then $u$ has convex choice. If $u$ strictly violates DSCD, then $u$ does not have convex choice.*

In light of its half-spaces interpretation, DSCD implying convex choice is straightforward. Conversely, a separating hyperplane theorem yields that under convex choice there is no strict violation of DSCD.[16]

Proposition 3 is closely related to a result of Grandmont (1978, p. 322). His condition (H.2) is a little stronger than imposing both convex choice and regular indifferences. Combined with a continuity condition—Grandmont's (H.1)—and an assumption that the convex type space is open, he obtains a characterization that is similar to, but more restrictive than, DSCD. See Appendix B.2 for details. The leading families of DSCD given above satisfy Grandmont's characterization. A one-dimensional example of DSCD that does not is $\Theta = (0, 1)$, $A = \{a', a''\}$, $u(a', \cdot) = 0$, $u(a'', \theta) = -1$ if $\theta < 1/2$ and $u(a'', \theta) = 1$ if $\theta \geq 1/2$; Grandmont's continuity requirement fails here.

If $\Theta \subset \mathbb{R}$, then DSCD is equivalent to Kartik et al.'s (2024b) "single-crossing differences" and, modulo indifferences, Proposition 3 is equivalent to part 1 of their Theorem 1. But in multiple dimensions the notions are distinct, and their result is not about convex choice. Similarly, Milgrom and Shannon's (1994) single-crossing preferences do not guarantee convex choice when $\Theta$ is multidimensional.[17]

[16] If $\Theta \subset \mathbb{R}$, then absent indifferences, a violation of DSCD is equivalent to a strict violation of DSCD. But not so in multiple dimensions. Consider the following two examples with $\Theta = [0, 1]^2$ and $A = \{a', a''\}$:




Both examples have no indifferences, a violation of DSCD, but no strict violation of DSCD. On the left, there is convex choice (so convex choice does not imply DSCD); on the right, convex choice fails (so no strict violation of DSCD does not imply convex choice).

[17] Neither does Barthel and Sabarwal's (2018) "$i$-single crossing property". Their notion is motivated by Quah's (2007) weakening of Milgrom and Shannon's (1994) comparisons of constraint/choice sets. Convex choice considers all choice sets.

McAfee and McMillan (1988) present a notion of "generalized single crossing". Their condition is formulated for a Euclidean allocation space and twice-differentiable utilities. For differentiable mechanisms, they establish that their condition implies that checking local IC along line segments is sufficient. Appendix B.3 provides an example demonstrating that McAfee and McMillan's condition does not imply convex choice, and hence does not imply DSCD. Consequently, as displayed explicitly in our example, there are non-differentiable mechanisms for which their condition does not yield sufficiency of local IC.

## 3.1. Convex Environments

In various settings, the agent may be choosing among lotteries. For instance, a mechanism designer may consider stochastic mechanisms; indeed, the revelation principle in general requires allowing for those. Or, the mechanism may take inputs from other agents with private information, so that from the interim point of view of one agent, her report induces a lottery. Alternatively, in a cheap-talk application, the receiver's decision may be determined by both the sender's message and the receiver's private preference type; Kartik et al. (2024b) detail this application with $\Theta, A \subset \mathbb{R}$.

Following Kartik et al. (2024b), we say that the environment $(A, \Theta, u)$ is *convex* if

$$\text{the set of functions } \{u(a, \cdot) : \Theta \to \mathbb{R}\}_{a \in A} \text{ is convex.} \qquad (\star)$$

That is, a convex environment has the property that for any pair of actions and any weighting, there is a third action that replicates the weighted sum of utilities of the original actions. Note that convexity is a property of the utility function rather than of $A$. However, if $A$ is convex, then it is straightforward that ($\star$) is assured by linearity of $u$ in its first argument. Expected utility over a convex set of lotteries (e.g., all lotteries) is thus a convex environment. In fact, rank-dependent expected utility also produces a convex environment (Kartik et al., 2024b, Example 2). An example without lotteries is $A = [\underline{a}_1, \overline{a}_1] \times \ldots \times [\underline{a}_k, \overline{a}_k] \subset \mathbb{R}^k$ and $u(a, \theta) = \sum_{i=1}^k g_i\left(a_{(i)}\right) f_i(\theta)$ with arbitrary functions $(f_i)_{i=1}^k$ and continuous functions $(g_i)_{i=1}^k$.

The following concepts are useful to elucidate the implications of DSCD in convex environments. We say that the utility function $\tilde{u}(a, \theta)$ *represents* $u(a, \theta)$ if $\tilde{u}(a, \theta) = h(u(a, \theta), \theta)$ for some function $h$ such that for all $\theta$, $h(\cdot, \theta) : \mathbb{R} \to \mathbb{R}$ is strictly increasing. This is simply the standard notion of (type-dependent) preference representation. An *affine representation* is any representation that is affine in $\theta$, i.e., has the form $v(a) \cdot \theta + w(a)$ for some $v : A \to \mathbb{R}^n$ and $w : A \to \mathbb{R}$. We say that $u$ is *one-dimensional* if there are $\alpha \in \mathbb{R}^n \setminus \{0\}$ and $\tilde{u} : A \times \mathbb{R} \to \mathbb{R}$

such that $u(a,\theta) \geq (>) u(a',\theta)$ if and only if $\tilde{u}(a, \alpha \cdot \theta) \geq (>) \tilde{u}(a', \alpha \cdot \theta)$.[18] In other words, any type $\theta$'s preferences are fully determined by the one-dimensional sufficient statistic $\alpha \cdot \theta$. A type $\theta$ is *totally indifferent* if $u(a,\theta) = u(a',\theta)$ for all $a, a' \in A$.

Regardless of a convex environment, any utility function with an affine representation satisfies DSCD. For, any affine utility clearly satisfies DSCD, and DSCD is an ordinal property that is preserved by any representation. If preferences are one-dimensional, then even in a convex environment there are DSCD utilities that do not have an affine representation: for example, $\Theta = [1,2]$, $A = \Delta(\{x_0, x_1, x_2\})$, and for any lottery $a \in A$, $u(a,\theta) = \sum_{i=0}^{2} a(x_i)\bar{u}(x_i,\theta)$ with $\bar{u}(x_0,\theta) = 0$, $\bar{u}(x_1,\theta) = \theta$ and $\bar{u}(x_2,\theta) = \theta^2$.[19] The following result says that under some additional assumptions, these two cases—one-dimensional preferences and those with an affine representation—exhaust DSCD in a convex environment.

**Proposition 4.** *Assume $\Theta = \mathbb{R}^n$, $u(a,\theta)$ is differentiable in $\theta$, and no type is totally indifferent.*[20] *If the environment is convex and $u$ satisfies DSCD, then either $u$ is one-dimensional or it has an affine representation.*

**Proof idea.** We pick an arbitrary action $a_*$ and normalize $u(a_*,\theta) = 0$ for all $\theta$. DSCD and the convex environment then imply that $u$ is linear-combinations DSC-preserving: for all finite index sets $I$, $\{a_i\}_{i\in I} \subset A$, and $\{\lambda_i\}_{i \in I} \subset \mathbb{R}$, it holds that $\sum_{i\in I} \lambda_i u(a_i,\cdot)$ is DSC.

Suppose that preferences are not one-dimensional. Then, there are actions $a_1, a_2 \in A$ such that $u_1(\cdot) := u(a_1,\cdot)$ and $u_2(\cdot) := u(a_2,\cdot)$ are not DSC in a common direction. So there must be two non-parallel hyperplanes such that $u_1$ vanishes on one and $u_2$ on the other. Since these hyperplanes must intersect and $\Theta = \mathbb{R}^n$, there is $\theta_0 \in \Theta$ with $u_i(\theta_0) = 0$ for $i = 1, 2$.

Now fix arbitrary $\theta_1 \in \Theta$ with $u_2(\theta_1) \neq 0$, and consider the function

$$u_3(\theta) := u_1(\theta) - \frac{u_1(\theta_1)}{u_2(\theta_1)} u_2(\theta).$$

Since $u_3$ is DSC and $u_3(\theta_0) = u_3(\theta_1) = 0$, it follows that $\nabla u_3(\theta_0)$ is orthogonal to $\theta_1 - \theta_0$.

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[18] For a DSCD preference, this is equivalent to the utility difference between every pair of actions satisfying DSC in the common direction $\alpha$.

[19] DSCD holds because the utility difference between any pair of lotteries is a linear combination of $\theta$ and $\theta^2$, which has at most one root in $\Theta$. To see that there is no affine representation, observe that because $u$ is an expected utility, any representation $\tilde{u}$ must be a type-dependent positive affine transformation of $u$, i.e., have the form $\tilde{u}(a,\theta) = b(\theta)u(a,\theta) + c(\theta)$ with $b(\cdot) > 0$. But no functions $b$ and $c$ can make both $b(\theta)\theta + c(\theta)$ and $b(\theta)\theta^2 + c(\theta)$ affine.

[20] As explained in the proof, instead of no totally indifferent type, it is enough to assume that for any type $\theta$ that is totally indifferent, there are two actions $a'$ and $a''$ such that $\nabla[u(a',\theta) - u(a'',\theta)] \neq 0$.

This implies $\left(\nabla u_1(\theta_0) - \frac{u_1(\theta_1)}{u_2(\theta_1)}\nabla u_2(\theta_0)\right) \cdot (\theta_1 - \theta_0) = 0$, which rearranges as

$$u_2(\theta_1) = u_1(\theta_1)\frac{\nabla u_2(\theta_0) \cdot (\theta_1 - \theta_0)}{\nabla u_1(\theta_0) \cdot (\theta_1 - \theta_0)}$$

so long as $\nabla u_1(\theta_0) \cdot (\theta_1 - \theta_0) \neq 0$. Hence, $u_2$ is essentially determined by $u_1$ and the vector $\nabla u_2(\theta_0)$, except possibly at $\theta_1$ such that either $u_2(\theta_1) = 0$ or $\nabla u_1(\theta_0) \cdot (\theta_1 - \theta_0) = 0$. Since the foregoing arguments can be applied to any $u(a, \theta)$ with $a \notin \{a_1, a_2\}$, it follows that $u(a_1, \cdot)$ largely determines the entire function $u : A \times \Theta \to \mathbb{R}$. In particular, if $u(a_1, \cdot)$ is affine then $u(a, \cdot)$ is affine for all $a$.

We can use the fact that $u(a_1, \cdot)$ is DSC with zero set given by a hyperplane (or empty or all of $\Theta$) to show that there is a representation $\tilde{u}$ such that $\tilde{u}(a_1, \cdot)$ is affine. Building on the arguments above, we can then show that $\tilde{u}(a, \cdot)$ is affine for all $a$. □

An expected utility is affine or one-dimensional if and only if its generating von Neumann-Morgenstern (vNM) utility is, respectively, affine or one-dimensional. Hence, for convex environments induced by lotteries and expected utility, Proposition 4 can be equivalently stated in terms of either the vNM or the expected-utility function.

Recall that DSCD essentially characterizes convex choice (Proposition 3), and we have argued that convex choice is crucial for tractability (Proposition 1) and appealing for other reasons (e.g., in cheap-talk games). We thus view Proposition 4 as suggesting that if one cannot substantially restrict the set of lotteries to consider (perhaps because of the presence of other agents), then in many contexts multidimensional expected-utility preferences without an affine representation will be unwieldy. This may shed light, in particular, on the prominence of the affine form in multidimensional mechanism design. It is generally difficult in those problems to rule out optimality of stochastic mechanisms, even under the affine form; see, e.g., Pycia (2006) and Manelli and Vincent (2007).[21] By contrast, there are canonical one-dimensional settings in which "regularity" assumptions do ensure that deterministic mechanisms are optimal (Strausz, 2006); in these cases, solutions have been obtained assuming only DSCD over deterministic allocations.[22]

Appendix B.1 shows that the assumption of $\Theta = \mathbb{R}^n$ cannot be dropped from Proposition 4. But one can obtain the result with alternative assumptions; what is important

[21] In some problems, progress has been made by identifying conditions ensuring that specific deterministic mechanisms of interest are optimal (e.g., Haghpanah and Hartline, 2021).

[22] Strausz's (2006) result is for a single-agent setting with transfers and quasi-linear utility. Beyond that, even one-dimensional mechanism design frequently assumes functional forms that assure DSCD over lotteries; Remark 3 details the relevant restrictions. Examples in the context of delegation include Amador and Bagwell (2020) and Kartik, Kleiner, and Van Weelden (2021).

is some richness, beyond the convexity requirement ($\star$), in terms of the family $\{u(a,\cdot)\}_{a\in A}$ when preferences are not one-dimensional. We offer one such variant below. Say that an action $a \in A$ *strictly dominates* $a' \in A$ if $u(a,\theta) > u(a',\theta)$ for all $\theta \in \Theta$. We also say that $u$ is *minimally rich* if there are no actions $a_0, a_1, a_2 \in A$ and functions $\lambda_0, \lambda_1 : A \to \mathbb{R}$ such that for all $a \in A$ and $\theta \in \Theta$,

$$u(a,\theta) - u(a_0,\theta) = \lambda_0(a)[u(a_1,\theta) - u(a_0,\theta)] + \lambda_1(a)[u(a_2,\theta) - u(a_0,\theta)].$$

That is, a failure of minimal richness means that after normalizing $u(a_0,\cdot) = 0$, it holds that for each action $a$, there is a linear combination such that all types' utilities from $a$ are the same linear combination of their utilities from $a_1$ and $a_2$.

**Proposition 5.** *Assume $u(a,\theta)$ is differentiable in $\theta$, minimally rich, has regular indifferences, and there is an action that strictly dominates another. If the environment is convex and $u$ satisfies DSCD, then either $u$ is one-dimensional or it has an affine representation.*

Proposition 5 substitutes Proposition 4's assumptions of $\Theta = \mathbb{R}^n$ and no total indifference with minimal richness, regular indifferences, and strict dominance between some pair of actions. Observe that the dominance assumption is satisfied whenever there is one component of the action space over which all types have strictly monotonic preferences; in particular, it holds in the quasi-linear environments common in mechanism design.

To illustrate how the "one-dimensional or affine representation" conclusion of Propositions 4 and 5 is useful, we return to the CES preferences discussed earlier. Consider $X, \Theta \subset \mathbb{R}^n_+$ each with nonempty interior, and the (generalized) CES utility $\bar{u}(x,\theta) = \left(\sum_{i=1}^n (x_{(i)})^r \theta_{(i)}\right)^s + w(x)$ with $r \in \mathbb{R}$ and $s > 0$. Although $\bar{u}$ satisfies DSCD, does the induced expected-utility function $u(a,\theta)$ over the lottery space $A = \Delta X$? In one dimension, $n = 1$, yes: for example, in that case any linear combination $\lambda\bar{u}(x,\theta) + \lambda'\bar{u}(x',\theta) = (\lambda x^{rs} + \lambda'(x')^{rs})\theta^s + (\lambda w(x) + \lambda' w(x'))$ is monotonic in $\theta$, hence (directionally) single crossing. But in multiple dimensions, $n > 1$, Proposition 5 implies that $u$ has DSCD if and only if $s = 1$; for, if $s \neq 1$, there is no affine representation of $u$.[23] As already mentioned, the case of $s = 1$ is prominent in multidimensional mechanism design.

[23] The assumptions on $\Theta$ and $X$ ensure that the assumptions of Proposition 5 are satisfied, and that preferences are not one-dimensional when $n > 1$. To see that there is no affine representation if $s \neq 1$, assume $0 \in X$ (to simplify the argument), fix some action $x_0 \neq 0$, and consider the representation

$$\bar{\bar{u}}(x,\theta) = [(x^r \cdot \theta)^s + w(x) - w(x_0)]\frac{x_0^r \cdot \theta}{(x_0^r \cdot \theta)^s}. \tag{2}$$

The function $\bar{\bar{u}}$ is affine in $\theta$ for $x = x_0$, but not for general $x$. For, if $x$ is not a scalar multiple of $x_0$, then changing $\theta$ on a hyperplane orthogonal to $x_0$ does not change the fraction in (2), and since $(x^r \cdot \theta)^s$ is not

Furthermore, since preferences are not changed by adding a function that depends only on type, the case of $s = 1$ subsumes quadratic-loss preferences, $\bar{u}(x, \theta) = -\|x - \theta\|^2$ with $\|\cdot\|$ the Euclidean norm. Interestingly, in their study of common-interest cheap talk with noisy communication—so that the sender's message induces a lottery over receiver decisions—Bauch (2024, Proposition 6.3) requires quadratic loss to obtain a convex-partitional equilibrium result. Without noise, Jäger et al. (2011, Corollary 1) could allow for preferences based on more general functions of the Euclidean norm.[24]

*Remark* 2. For convex environments, Kartik et al. (2024b) provide a characterization of preferences that satisfy their single-crossing differences (SCD). Our message that multidimensional preferences must have an affine representation does not obtain under SCD, because SCD corresponds to interval choice whereas DSCD to convex choice. To illustrate, consider expected utility over lotteries on a binary outcome space with vNM preferences given by the right figure of footnote 4. With only two outcomes, expected utility has SCD/interval choice or DSCD/convex choice if and only if the vNM utility has the respective property. Hence, this is an example of expected utility with SCD but not DSCD; indeed, because of the figure's non-linear indifference curve, there is no affine representation nor are the preferences one-dimensional.[25]

*Remark* 3. Although one-dimensional preferences with DSCD in a convex environment need not have an affine representation, their structure can be characterized using Kartik et al.'s (2024b) results. If $\Theta$ is compact, then all types' preferences must be a convex combination of two "extreme" types', with an ordered weighting. More precisely, there must be a representation $\lambda(\alpha \cdot \theta)\overline{u}(a) + (1 - \lambda(\alpha \cdot \theta))\, \underline{u}(a)$, where $\alpha \in \mathbb{R}^n \setminus \{0\}$ is the direction in which preferences are one-dimensional, $\lambda : \mathbb{R} \to [0, 1]$ is an increasing function, and $\overline{u}$ and $\underline{u}$ represent the preferences of the types that respectively maximize and minimize $\alpha \cdot \theta$.

affine when $s \neq 1$, $\bar{\bar{u}}(x, \cdot)$ is not affine. Since any representation of $\bar{u}$ is a type-dependent positive affine transformation of $\bar{\bar{u}}$, there is no affine representation.

[24] Cheap-talk aficionados may note that Blume, Board, and Kawamura (2007) get interval equilibria in their one-dimensional model of noisy communication without imposing quadratic loss on the sender's utility. That owes to their special "truth-or-noise" garbling: either the sender's message goes through correctly, or a random message is drawn from an exogenous distribution. So when choosing her message, the sender can condition on the no-noise event, effectively choosing among deterministic receiver actions. In a multidimensional setting, including Bauch's (2024), truth-or-noise garbling would ensure that all equilibria are convex partitional (modulo details about indifferences) if the sender's utility has DSCD, even if it does not have DSCD over lotteries.

[25] While neither Proposition 4's nor Proposition 5's hypotheses are satisfied in this example, one can modify it to make the same point while satisfying either proposition's hypotheses. We also note that the example in footnote 4's left figure naturally extends to illustrate how expected utility can have an affine representation (hence DSCD) but not SCD.

# 4. Conclusion

Convex choice is a valuable property. We have shown that (i) in a suitable sense, it characterizes the sufficiency of local IC (Proposition 1), (ii) it speaks to the implementability of allocation rules (Proposition 2) and is relevant to applications like cheap talk, (iii) it is essentially equivalent to a form of single crossing with a simple geometric interpretation (Proposition 3), and (iv) in convex environments satisfying some regularity conditions, it reduces to "one-dimensional or affine representation" (Propositions 4 and 5). Convex choice has also been used in other contexts, such as preference aggregation (Grandmont, 1978; Caplin and Nalebuff, 1988) and social learning (Kartik, Lee, Liu, and Rappoport, 2024a).

An alternative notion—generally weaker and equivalent only in one dimension—is connected choice: the set of types that find an action optimal is connected. As mentioned after Proposition 1 and elaborated in Appendix A.7, connected choice characterizes a weaker form of the sufficiency of local IC. Nevertheless, we find convex choice more appealing for three (related) reasons. First, it ensures that IC between any two types can be verified via local IC along that line segment; this is more tractable than checking local IC along all paths connecting the two types. Indeed, for a related reason, convex choice cannot be replaced by connected choice in Proposition 2. Second, convex choice sets are typically more economically meaningful; for instance, in multidimensional cheap talk, equilibria that merely partition the sender's type space into connected sets would be less satisfactory. Third, convex choice ties to directional single crossing and its implications/tractability in a way that connected choice is not amenable to.

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# Appendix A: Main Appendix

## A.1. Proof of Proposition 1

**Proof that statement 1 $\implies$ statement 2.** Assume convex choice and regular indifferences. Fix an arbitrary line segment $\Theta' \subset \Theta$ and let $m : \Theta' \to A$ satisfy local IC. Fix any two distinct types in $\Theta'$; we can label the line segment between these types as $[0,1]$, corresponding to the convex combinations of the two types. Our goal is to show that $u(m(0),0) \geq u(m(1),0)$, which implies IC on $\Theta'$.

Local IC implies that each type $\theta \in [0,1]$ has a neighborhood $N_\theta \subset [0,1]$ satisfying condition (LIC). For each $\theta$, there is an open ball $B_\theta \subset N_\theta$ centered at $\theta$. These balls form an open cover of $[0,1]$; as $[0,1]$ is compact, there is a finite subcover $\{B_{\theta_i}\}_{i=1}^m$. We assume without loss that this subcover is minimal and its indices satisfy $\theta_1 < \ldots < \theta_m$.

Then $0 \in B_{\theta_1}$ and $1 \in B_{\theta_m}$. Local IC implies

$$u(m(0), 0) \geq u(m(\theta_1), 0). \tag{3}$$

Now choose any $\tilde{\theta}_1 \in B_{\theta_1} \cap B_{\theta_2}$ with $\tilde{\theta}_1 > \theta_1$. By local IC,

$$u(m(\theta_1), \theta_1) \geq u(m(\tilde{\theta}_1), \theta_1), \tag{4}$$

$$u(m(\tilde{\theta}_1), \tilde{\theta}_1) \geq u(m(\theta_1), \tilde{\theta}_1). \tag{5}$$

We claim that

$$u(m(\theta_1), 0) \geq u(m(\tilde{\theta}_1), 0). \tag{6}$$

To prove inequality (6), suppose to the contrary $u(m(\tilde{\theta}_1), 0) > u(m(\theta_1), 0)$. Then inequality (4) implies $\theta_1 > 0$. If inequality (5) holds strictly, then convex choice implies $u(m(\tilde{\theta}_1), \theta_1) > u(m(\theta_1), \theta_1)$, contradicting (4). If (5) holds with equality, then since $\theta_1 \in (0, \tilde{\theta}_1)$, it follows from regular indifferences and convex choice that $u(m(\tilde{\theta}_1), \theta_1) > u(m(\theta_1), \theta_1)$, contradicting (4).

Combining (3) and (6), $u(m(0), 0) \geq u(m(\tilde{\theta}_1), 0)$. Continuing in the same fashion, we obtain $u(m(0), 0) \geq u(m(1), 1)$, as desired. Q.E.D.

**Proof that statement 2 $\Longrightarrow$ statement 1.** We prove the contrapositive, first for convex choice and then for regular indifferences.

If $u$ violates convex choice, then there are $a', a'' \in A$ and $\theta_1, \theta_2 \in \Theta$ and $\theta' \in \ell(\theta_1, \theta_2)$ such that $u(a'', \theta_i) > u(a', \theta_i)$ for $i = 1, 2$ and $u(a'', \theta') \leq u(a', \theta')$. Let $\Theta' := \ell(\theta_1, \theta_2)$ and let $a^*(\theta)$ be an arbitrary selection from $\arg\max_{a \in \{a', a''\}} u(a, \theta)$. Consider the mechanism $m : \Theta' \to \{a', a''\}$ given by

$$m(\theta) := \begin{cases} a^*(\theta) & \text{if } \theta \in \ell(\theta_1, \theta') \setminus \{\theta'\} \\ a' & \text{otherwise.} \end{cases}$$

This mechanism is locally IC because all types in $\ell(\theta_1, \theta')$ are getting an optimal action from $\{a', a''\}$ and the mechanism is constant on $\ell(\theta', \theta_2)$. But IC fails because type $\theta_2$ can profitably mimic $\theta_1$.

If $u$ violates regular indifferences, then there are $a', a'' \in A$ and $\theta_1, \theta_2 \in \Theta$ and $\theta' \in \ell(\theta_1, \theta_2) \setminus \{\theta_1, \theta_2\}$ such that $u(a', \theta_1) = u(a'', \theta_1)$, $u(a'', \theta_2) > u(a', \theta_2)$, and $u(a', \theta') \geq u(a'', \theta')$. Consider the same mechanism $m$ defined in the previous paragraph, except that $m(\theta_1) := a''$. It is locally IC but not IC by the argument given in the previous paragraph. Q.E.D.

## A.2. Proof of Proposition 2

Let $\nu : \Theta \to Y$ be a weakly monotone allocation rule with finite range. Since $\Theta \subset \mathbb{R}^n$ is convex and $\tilde{u}(y, \theta)$ is continuous in $\theta$, Theorem 5 in Berger et al. (2017, p. 380) implies that it is sufficient to show that $\nu$ is "line implementable" (Berger et al., 2017, p. 373).

Towards contradiction, suppose there is a line segment such that the restriction of $\nu$ to this line segment is not implementable. We can label types on this line segment as $[0, 1]$ corresponding to the convex combinations of its end points. It follows from Rochet (1987) that cyclical monotonicity is violated: there is a finite set $\Theta' \equiv \{\theta_1, ..., \theta_K\} \subset [0, 1]$ such that

$$\sum_{i=1}^{K} [\tilde{u}(\nu(\theta_{i+1}), \theta_i) - \tilde{u}(\nu(\theta_i), \theta_i)] > 0,$$

where $\theta_{K+1} := \theta_1$, and the restriction of $\nu$ to $\Theta'$ is not implementable.

Relabel indices so that $\theta_i \leq \theta_{i+1}$ and define transfers on $\Theta'$ by $t(\theta_1) = 0$ and $t(\theta_i) = \tilde{u}(\nu(\theta_i), \theta_i) - \tilde{u}(\nu(\theta_{i-1}), \theta_i) + t(\theta_{i-1})$ for $i = 2, \ldots, K$. Hence, no type $\theta_i$ has a profitable deviation to $\theta_{i+1}$. Moreover, weak monotonicity implies that $\tilde{u}(\nu(\theta_i), \theta_i) - t(\theta_i) \geq \tilde{u}(\nu(\theta_{i+1}), \theta_i) - t(\theta_{i+1})$. Hence, no type $\theta_i$ has a profitable deviation to $\theta_{i-1}$.

Since the restriction of $\nu$ to $\Theta'$ is not implementable, the mechanism $m \equiv (\nu, t)$ defined on $\Theta'$ is not IC: there are $\theta_i$ and $\theta_j$ such that $\theta_i$ strictly prefers $m(\theta_j)$ to $m(\theta_i)$. Without loss, we can assume $j > i$ and $\theta_i$ weakly prefers $m(\theta_i)$ to $m(\theta_{j-1})$. Hence,

$$u(m(\theta_j), \theta_i) > u(m(\theta_{j-1}), \theta_i), \tag{7}$$

$$u(m(\theta_j), \theta_{j-1}) \leq u(m(\theta_{j-1}), \theta_{j-1}), \text{ and} \tag{8}$$

$$u(m(\theta_j), \theta_j) \geq u(m(\theta_{j-1}), \theta_j). \tag{9}$$

These inequalities contradict $u$ having convex choice and regular indifferences.[26] Q.E.D.

[26] Inequalities (7) and (9) and regular indifferences imply that some type in $\ell(\theta_{j-1}, \theta_j) \setminus \{\theta_{j-1}\}$ strictly prefers $m(\theta_j)$ to $m(\theta_{j-1})$. In light of (7) and (8), that violates convex choice.

## A.3. Proof of Remark 1

It suffices to prove that $\|\cdot\|$ is an inner-product norm, as standard arguments then imply that it is weighted Euclidean.

Accordingly, suppose to contradiction that $\|\cdot\|$ is not an inner-product norm yet there is DSCD. For any pair of actions $a, a' \in A$, the set of indifferent types is the set of types that are equidistant to $a$ and $a'$, i.e., $\{\theta \in \Theta : \|a - \theta\| = \|a' - \theta\|\}$. DSCD implies that this set is convex. However, as $\|\cdot\|$ is not an inner-product norm, it follows from Panda and Kapoor (1974, Corollary 1.3) that for some $a \in \mathbb{R}^n$, the set $\{\theta \in \mathbb{R}^n : \|a - \theta\| = \|-a - \theta\|\}$ is not convex. Hence, there exist $\theta, \theta' \in \mathbb{R}^n$ and $\lambda \in (0, 1)$ such that $\theta$ and $\theta'$ are equidistant to $a$ and $-a$, but $\theta'' := \lambda\theta + (1 - \lambda)\theta'$ is closer to, say, $a$. Let $\theta_0$ be in the interior of $A \cap \Theta$. We can then choose $\varepsilon > 0$ sufficiently small such that

$$\theta_0 + \varepsilon\theta, \theta_0 + \varepsilon\theta', \theta_0 + \varepsilon\theta'', \theta_0 + \varepsilon a, \theta_0 - \varepsilon a \in A \cap \Theta$$

and obtain

$$\begin{aligned} u(\theta_0 + \varepsilon a, \theta_0 + \varepsilon\theta'') &= -l(\varepsilon\|a - \theta''\|) > -l(\varepsilon\|-a - \theta''\|) = u(\theta_0 - \varepsilon a, \theta_0 + \varepsilon\theta''), \\ u(\theta_0 + \varepsilon a, \theta_0 + \varepsilon\theta) &= u(\theta_0 - \varepsilon a, \theta_0 + \varepsilon\theta), \\ u(\theta_0 + \varepsilon a, \theta_0 + \varepsilon\theta') &= u(\theta_0 - \varepsilon a, \theta_0 + \varepsilon\theta'), \end{aligned}$$

contradicting that the set of indifferent types is convex. Q.E.D.

## A.4. Proof of Proposition 3

Recall that $D_{a,a'}(\theta) \equiv u(a, \theta) - u(a', \theta)$.

For the proposition's first statement, suppose $u$ has DSCD. It is sufficient to show convex choice for all binary choice sets. So consider any $A' = \{a', a''\} \subset A$. Let $\theta', \theta'' \in \{\theta : \{a'\} = \arg\max_{a \in A'} u(a, \theta)\}$. We have $D_{a',a''}(\theta') > 0$ and $D_{a',a''}(\theta'') > 0$. Since $u$ has DSCD, there is $\alpha \in \mathbb{R}^n \setminus \{0\}$ such that $D_{a',a''}(\cdot)$ is DSC in direction $\alpha$. This implies $D_{a',a''}(\theta) > 0$ for all $\theta \in \ell(\theta', \theta'')$, as required.

For the proposition's second statement, we prove the contrapositive. Suppose $u$ has convex choice. Fix arbitrary $a', a'' \in A$ and let

$$\begin{aligned} \Theta' &:= \{\theta : D_{a',a''}(\theta) > 0\}, \\ \Theta'' &:= \{\theta : D_{a',a''}(\theta) < 0\}. \end{aligned}$$

It is straightforward that there is not a strict violation of DSCD if either of these sets is empty. So assume $\Theta' \neq \emptyset$ and $\Theta'' \neq \emptyset$. Then $\Theta'$ and $\Theta''$ are nonempty, convex (by convex choice), and disjoint sets. A standard separating hyperplane theorem implies that there is $\alpha \in \mathbb{R}^n \setminus \{0\}$ such that $\alpha \cdot \theta' \leq \alpha \cdot \theta''$ for all $\theta' \in \Theta'$ and $\theta'' \in \Theta''$. Hence, there are no $\theta, \theta', \theta'' \in \Theta$ with $\alpha \cdot \theta < \alpha \cdot \theta' < \alpha \cdot \theta''$ and $\min\{D_{a_1,a_2}(\theta), D_{a_1,a_2}(\theta'')\} > 0 > D_{a_1,a_2}(\theta')$, i.e., there is not a strict violation of DSCD. Q.E.D.

## A.5. Proof of Proposition 4

For this proof, we fix an arbitrary action $a_* \in A$ and work with the function

$$f(a, \theta) := u(a, \theta) - u(a_*, \theta).$$

Since we are interested in representations of $u$, we say that a function $\tilde{f}$ is a *representative* of $f$ if there is a representation $\tilde{u}$ of $u$ such that $\tilde{f}(a, \theta) = \tilde{u}(a, \theta) - \tilde{u}(a_*, \theta)$. Note that any type-dependent positive linear transformation of $f$ (i.e., $g(\theta)f(a, \theta)$ with $g(\cdot) > 0$) is a representative of $f$. We say that $f$ is *DSC-preserving* if for all finite index sets $I$, $\{a_i\}_{i\in I} \subset A$, and $\{\lambda_i\}_{i\in I} \subset \mathbb{R}$, it holds that $\sum_{i\in I} \lambda_i f(a_i, \cdot)$ is DSC.

**Proof of Proposition 4.** Let $A_* := A \setminus \{a_*\}$. Analogous to Kartik et al. (2024b, proof of Theorem 2), for every $\mu : A_* \to \mathbb{R}$ with finite support $A_\mu$, there exist probability distributions $P, Q$ on $A_\mu \cup \{a_*\}$ such that $\sum_{a\in A_\mu} f(a, \theta)\mu(a)$ is a multiple of

$$\sum_{a\in A_\mu\cup\{a_*\}} u(a, \theta)P(a) - \sum_{a\in A_\mu\cup\{a_*\}} u(a, \theta)Q(a). \tag{10}$$

Since the environment is convex and $u$ has DSCD, the function (10) is DSC, which implies that $\sum_{a\in A_\mu} f(a, \theta)\mu(a)$ is also DSC. Hence, $f(a, \theta)$ is DSC-preserving. Proposition 6 below implies that $f$ has an affine representative or is one-dimensional. It follows that $u$ has an affine representation or is one-dimensional. Q.E.D.

Accordingly, the key to Proposition 4 is the following result.

**Proposition 6.** *Assume $\Theta = \mathbb{R}^n$ and $f(a, \theta)$ is DSC-preserving and differentiable in $\theta$. If no type is totally indifferent, then $f$ is either one-dimensional or has an affine representative.*

The proof of Proposition 6 is lengthy, so we provide an outline first.

**Outline of the proof of Proposition 6.** Suppose that $f$ is not one dimensional. Lemma 1 establishes that there are actions $a_0$ and $a_1$ such that $f(a_0, \cdot)$ and $f(a_1, \cdot)$ are not DSC in a common direction. Lemma 2 shows that the zero sets of the two functions are non-parallel hyperplanes (or empty). Consequently, since $\Theta = \mathbb{R}^n$, there is $\theta_0$ such that $f(a_0, \theta_0) = f(a_1, \theta_0) = 0$. We argue that $f(a_1, \cdot)$ is essentially determined by $f(a_0, \cdot)$ and $\nabla f(a_1, \theta_0)$ (Lemma 3), and use this insight to conclude that under our assumptions, $f$ has an affine representative (Lemma 4).

**Lemma 1.** *Let $\alpha \in \mathbb{R}^n \setminus \{0\}$ and $f : A \times \Theta \to \mathbb{R}$. If $f(a, \cdot)$ is DSC in direction $\alpha$ for all $a \in A$, then $f$ is one-dimensional.*

**Proof.** For any $x \in \mathbb{R}$, choose any $\theta_x \in \mathbb{R}^n$ such that $\alpha \cdot \theta_x = x$ and define, for all $a \in A$, $\tilde{f}(a, \cdot) : \mathbb{R} \to \mathbb{R}$ by $x \mapsto f(a, \theta_x)$.

We claim that $\operatorname{sign}(f(a, \theta)) = \operatorname{sign}(\tilde{f}(a, \alpha \cdot \theta))$ for all $\theta$ and $a$. Let $x = \alpha \cdot \theta$ and observe that $\tilde{f}(a, \alpha \cdot \theta) = \tilde{f}(a, \alpha \cdot \theta_x)$. Moreover, since $f(a, \cdot)$ is DSC in direction $\alpha$, we obtain

$$\operatorname{sign}(f(a, \theta)) = \operatorname{sign}(f(a, \theta_x)) = \operatorname{sign}(\tilde{f}(a, \alpha \cdot \theta_x)) = \operatorname{sign}(\tilde{f}(a, \alpha \cdot \theta)). \qquad \text{Q.E.D.}$$

**Lemma 2.** *Assume $\Theta = \mathbb{R}^n$, $f$ is DSC-preserving, and for some $a_1, a_2 \in A$, $f(a_1, \cdot)$ and $f(a_2, \cdot)$ are not DSC in a common direction. Then, for $i = 1, 2$, $\{\theta : f(a_i, \theta) = 0\}$ is a hyperplane or empty.*

**Proof.** For $i = 1, 2$, it is sufficient to prove that the zero set of $f(a_i, \cdot)$ is contained in a hyperplane, since $f(a_i, \cdot)$ being DSC then implies that its zero set is a hyperplane or empty. Moreover, there is no loss of generality in proving it for just $f(a_1, \cdot)$.

Accordingly, suppose towards contradiction that the zero set of $f(a_1, \cdot)$ is not contained in a hyperplane. Since $f(a_1, \cdot)$ and $f(a_2, \cdot)$ are not DSC in a common direction, there exist $\theta_0$ and $\theta_1$ with $f(a_i, \theta_0) = 0$ and $f(a_i, \theta_1) \neq 0$ for $i = 1, 2$. Moreover, one can choose $\theta_0$ and $\theta_1$ with these properties such that there is $\theta' \in \ell(\theta_0, \theta_1)$ with $f(a_1, \theta') = 0 \neq f(a_2, \theta')$; see Figure 3 for an illustration. Then the linear combination $f_3(\theta) := f(a_1, \theta) - \frac{f(a_1, \theta_1)}{f(a_2, \theta_1)} f(a_2, \theta)$ satisfies $f_3(\theta_0) = f_3(\theta_1) = 0 \neq f_3(\theta')$. Since $\theta' \in \ell(\theta_0, \theta_1)$, the function $f_3$ is not DSC, which contradicts $f$ being DSC-preserving. Q.E.D.

**Lemma 3.** *Let $f$ be DSC-preserving. Let $a_0, a_1 \in A$ be such that $f(a_0, \cdot)$ and $f(a_1, \cdot)$ are not DSC in a common direction and suppose $\theta_0$ satisfies $f(a_0, \theta_0) = f(a_1, \theta_0) = 0$.*

*If $f(a_0, \cdot)$ and $f(a_1, \cdot)$ are differentiable at $\theta_0$ and $\nabla f(a_0, \theta_0) \neq 0$, then for all $\theta$ with*

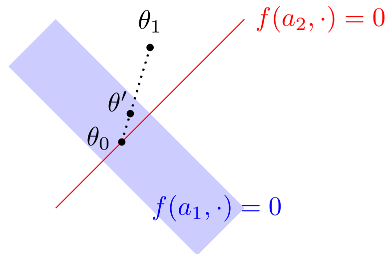


**Figure 3:** Illustration for Lemma 2

*$f(a_0, \theta), f(a_1, \theta) \neq 0$, it holds that*

$$f(a_1, \theta) = f(a_0, \theta) \frac{\nabla f(a_1, \theta_0) \cdot (\theta - \theta_0)}{\nabla f(a_0, \theta_0) \cdot (\theta - \theta_0)}.$$

**Proof.** Let $f_i(\theta) := f(a_i, \theta)$ for all $\theta$ and $i = 0, 1$. Fix arbitrary $\theta \in \Theta$ with $f_0(\theta) \neq 0$ and $f_1(\theta) \neq 0$, and consider the function

$$f_2(\theta') := f_0(\theta') - \frac{f_0(\theta)}{f_1(\theta)} f_1(\theta').$$

Since $f$ is DSC-preserving, $f_2$ is DSC. Moreover, $f_2(\theta_0) = f_2(\theta) = 0$ and, therefore,

$$\nabla f_2(\theta_0) \cdot (\theta - \theta_0) = \left[ \nabla f_0(\theta_0) - \frac{f_0(\theta)}{f_1(\theta)} \nabla f_1(\theta_0) \right] \cdot (\theta - \theta_0) = 0.$$

Moreover, $\nabla f_0(\theta_0) \cdot (\theta - \theta_0) \neq 0$ (because $\nabla f_0(\theta_0) \neq 0$ and $f_0(\theta) \neq 0$), hence $\nabla f_1(\theta_0) \cdot (\theta - \theta_0) \neq 0$. Therefore, for all $\theta$ with $f_0(\theta) \neq 0$ and $f_1(\theta) \neq 0$,

$$f_1(\theta) = f_0(\theta) \frac{\nabla f_1(\theta_0) \cdot (\theta - \theta_0)}{\nabla f_0(\theta_0) \cdot (\theta - \theta_0)}.$$ Q.E.D.

**Lemma 4.** *Assume $f$ is DSC-preserving and, for all $a \in A$, $f(a, \cdot)$ is differentiable with a zero set that is either a hyperplane (intersected with $\Theta$), $\Theta$, or empty.*

*If there are $a_0, a_1, a_2 \in A$ and $\theta_0 \in \Theta$ such that $f(a_0, \theta_0) = f(a_1, \theta_0) = 0 \neq f(a_2, \theta_0)$ and $f(a_0, \cdot)$ and $f(a_1, \cdot)$ are not DSC in a common direction, then $f$ has an affine representative.*

**Proof.** Let $f_i(\theta) := f(a_i, \theta)$ for $i = 0, 1$.

**Case 1:** Suppose $\nabla f_0(\theta_0) \neq 0$.

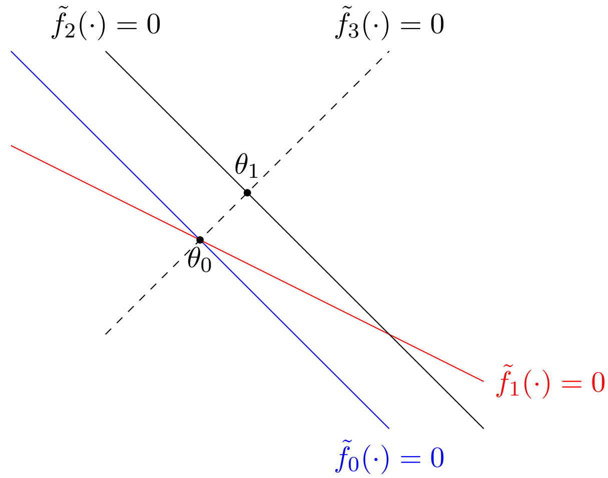


**Figure 4:** A step in the proof of Lemma 4.

By Lemma 3, for all $\theta$ with $f_0(\theta) \neq 0$ and $f_1(\theta) \neq 0$, we have

$$f_1(\theta) = f_0(\theta)\frac{\nabla f_1(\theta_0)\cdot(\theta-\theta_0)}{\nabla f_0(\theta_0)\cdot(\theta-\theta_0)}. \tag{11}$$

For each $i = 0, 1$, the set $\{\theta : f_i(\theta) = 0\}$ is neither empty (as it contains $\theta_0$) nor $\Theta$ (as $f_0$ and $f_1$ are not DSC in a common direction), and therefore, by hypothesis, is a hyperplane. Moreover, for each $i = 0, 1$, since $f_i$ is DSC and $\nabla f_i(\theta_0) \neq 0$, it follows that $\operatorname{sign}(f_i(\theta)) = \operatorname{sign}(\nabla f_i(\theta_0)\cdot(\theta-\theta_0))$. Therefore, we can define a representative $\tilde f$ as follows:

$$\tilde f(a,\theta) := \begin{cases} f(a,\theta)\frac{\nabla f_0(\theta_0)\cdot(\theta-\theta_0)}{f_0(\theta)} & \text{if } f_0(\theta)\neq 0 \\ f(a,\theta)\frac{\nabla f_1(\theta_0)\cdot(\theta-\theta_0)}{f_1(\theta)} & \text{if } f_0(\theta) = 0 \neq f_1(\theta) \\ kf(a,\theta) & \text{if } f_0(\theta) = f_1(\theta) = 0, \end{cases} \tag{12}$$

for some $k \in \mathbb{R}$ that remains to be specified. It follows from (11) and (12) that for each $i = 0, 1$, the function $\tilde f(a_i, \cdot)$ is affine, as it is 0 on $\{\theta : f_0(\theta) = f_1(\theta) = 0\}$ and $\nabla f_i(\theta_0)\cdot(\theta-\theta_0)$ otherwise.

Consider any $a_2 \in A$ and let $f_2 := f(a_2, \cdot)$ and $\tilde f_2 := \tilde f(a_2, \cdot)$. We will argue that $\tilde f_2$ is affine, which proves the lemma.

- First, suppose $\tilde f_2(\theta_0) \neq 0$ and $\tilde f_2$ has a nonempty zero set. By DSC, the zero set of $\tilde f_2$ contains a hyperplane (that does not pass through $\theta_0$). Since the zero sets of $\tilde f_0$ and $\tilde f_1$ are hyperplanes (that pass through $\theta_0$), there is $\theta_1$ with $\tilde f_2(\theta_1) = 0$, $\tilde f_0(\theta_1) \neq 0$ and

$\tilde{f}_1(\theta_1) \neq 0$. Moreover,

$$\tilde{f}_3(\theta) := \tilde{f}_0(\theta) - \frac{\tilde{f}_0(\theta_1)}{\tilde{f}_1(\theta_1)}\tilde{f}_1(\theta)$$

is affine as a linear combination of affine functions and satisfies $\tilde{f}_3(\theta_1) = \tilde{f}_3(\theta_0) = 0$. Since $\tilde{f}_2(\theta_1) = 0 \neq \tilde{f}_2(\theta_0)$, $\tilde{f}_2$ and $\tilde{f}_3$ are not DSC in a common direction. See Figure 4. Since $f_0(\theta_1) \neq 0$ and $f_0$ is continuous, we observe from (12) that in a neighborhood of $\theta_1$,

$$\tilde{f}_2(\theta) = f_2(\theta)\frac{\nabla f_0(\theta_0)\cdot(\theta-\theta_0)}{f_0(\theta)}.$$

Since $f_2$ and $f_0$ are differentiable, $\tilde{f}_2$ is differentiable at $\theta_1$ as a composition of differentiable functions. Also, $\tilde{f}$ is DSC-preserving because $f$ is DSC-preserving and $\tilde{f}$ is a type-dependent positive affine transformation of $f$.[27] By Lemma 3,

$$\tilde{f}_2(\theta) = \tilde{f}_3(\theta)\frac{\nabla \tilde{f}_2(\theta_1)\cdot(\theta-\theta_1)}{\nabla \tilde{f}_3(\theta_1)\cdot(\theta-\theta_1)}$$

on the set $\{\theta : f_2(\theta) \neq 0 \text{ and } f_3(\theta) \neq 0\}$. Since $\tilde{f}_3$ is affine, it follows that $\tilde{f}_2$ is affine on that set. Lemma 2 implies that the zero set of $\tilde{f}_2$ is a hyperplane; hence, $\tilde{f}_2$ is affine except possibly on $\{\theta : \tilde{f}_3(\theta) = 0\}$ (which is a hyperplane) and one can check that there is a choice of $k$ in (12) that makes $\tilde{f}_2$ affine everywhere.

- Next, suppose $\tilde{f}_2(\theta_0) \neq 0$ and $\tilde{f}_2$ is strictly positive (or strictly negative) on $\Theta$. Fix $\theta_2$ with $\tilde{f}_0(\theta_2) \neq 0$ and define $\tilde{f}_3(\theta) := \tilde{f}_2(\theta) - \frac{\tilde{f}_2(\theta_0)}{\tilde{f}_0(\theta_2)}\tilde{f}_0(\theta)$. Then $\tilde{f}_3(\theta_2) = 0$. If $\tilde{f}_3$ vanishes everywhere, $\tilde{f}_2$ is affine. Otherwise, the arguments in the previous bullet point imply that $\tilde{f}_3$ is affine. Since $\tilde{f}_0$ is affine, it follows that $\tilde{f}_2$ is affine.
- It remains to consider $\tilde{f}_2(\theta_0) = 0$. Lemma 3 yields that for all $\theta$ with $f_0(\theta) \neq 0$ and $f_2(\theta) \neq 0$, we have

$$f_2(\theta) = f_0(\theta)\frac{\nabla f_2(\theta_0)\cdot(\theta-\theta_0)}{\nabla f_0(\theta_0)\cdot(\theta-\theta_0)}$$

and, therefore, $\tilde{f}_2(\theta) = \nabla f_2(\theta_0)\cdot(\theta-\theta_0)$. Similarly, for all $\theta$ with $f_0(\theta) = 0$ and

[27] Indeed, fix arbitrary $a, a' \in A$ and $\lambda, \lambda' \in \mathbb{R}$. Then $\lambda f(a,\theta) + \lambda' f(a',\theta)$ is DSC and, for all $\theta$,

$$\text{sign}\left(\lambda f(a,\theta) + \lambda' f(a',\theta)\right) = \text{sign}\left(\lambda\beta(\theta) f(a,\theta) + \lambda'\beta(\theta) f(a',\theta)\right) = \text{sign}\left(\lambda \tilde{f}(a,\theta) + \lambda' \tilde{f}(a',\theta)\right),$$

where $\beta(\theta) > 0$ as defined in (12). Therefore, $\lambda\tilde{f}(a,\theta) + \lambda'\tilde{f}(a',\theta)$ is DSC. We remark that in general, type-dependent monotonic transformations (rather than positive affine transformations) of $f$ need not be DSC-preserving.

$f_1(\theta) \neq 0$ and $f_2(\theta) \neq 0$, we have

$$f_2(\theta) = f_1(\theta) \frac{\nabla f_2(\theta_0) \cdot (\theta - \theta_0)}{\nabla f_1(\theta_0) \cdot (\theta - \theta_0)}$$

and, therefore, $\tilde{f}_2(\theta) = \nabla f_2(\theta_0) \cdot (\theta - \theta_0)$. It follows that $\tilde{f}_2$ is affine.

**Case 2:** Suppose $\nabla f_0(\theta_0) = 0$.

Since we assumed that no type is totally indifferent, there is $a' \in A$ with $f(a', \theta_0) \neq 0$.[28] By choosing a different representative if necessary, we can assume without loss of generality that $f(a', \cdot)$ is constant in a neighborhood of $\theta_0$ (while maintaining differentiability of $f(a, \cdot)$ for all $a \in A$). Because every neighborhood of $\theta_0$ contains $\theta_0'$ with $\nabla f_0(\theta_0') \neq 0$ (otherwise $f_0$ would be identically zero in an neighborhood of $\theta_0$), there are $\theta_0' \in \Theta$ and $\lambda_0, \lambda_1 \in \mathbb{R}$ such that $f_0(\theta_0') + \lambda_0 f(a', \theta_0') = 0$, $f_1(\theta_0') + \lambda_1 f(a', \theta_0') = 0$, and $\nabla f_0(\theta_0') + \lambda_0 \nabla f(a', \theta_0') \neq 0$. The arguments for Case 1 then establish that $f$ has an affine representative. Q.E.D.

**Proof of Proposition 6.** If there is $\alpha \in \mathbb{R}^n \setminus \{0\}$ such that $f(a, \cdot)$ is DSC in direction $\alpha$ for all $a \in A$, then preferences are one-dimensional (see Lemma 1). So suppose there are $a_0, a_1 \in A$ such that $f(a_0, \cdot)$ and $f(a_1, \cdot)$ are not DSC in a common direction, and define $f_i(\theta) := f(a_i, \theta)$ for $i = 0, 1$. Using Lemma 2, for $i = 0, 1$ the zero set of $f_i$ is a hyperplane.[29] Because $f_0(\cdot)$ and $f_1(\cdot)$ are not DSC in a common direction, the hyperplanes on which each function vanishes intersect. Hence, since $\Theta = \mathbb{R}^n$, there is $\theta_0$ with $f_0(\theta_0) = f_1(\theta_0) = 0$. By assumption, there is $a_2 \in A$ such that $f(a_2, \theta_0) \neq 0$ and it follows from Lemma 4 that $f$ has an affine representative. Q.E.D.

## A.6. Proof of Proposition 5

Let action $a^*$ strictly dominate action $a_*$ and define $f(a, \theta) := \frac{u(a,\theta) - u(a_*,\theta)}{u(a^*,\theta) - u(a_*,\theta)}$. Note that $f(a_*, \cdot) = 0$, $f(a^*, \cdot) = 1$, and $f(a, \theta)$ is differentiable in $\theta$. Since $u$ has regular indifferences, the zero set of any linear combination of $f(a, \cdot)$'s is either a hyperplane (intersected with $\Theta$),

[28] Under the weaker assumption stated in footnote 20, if there is no such action (i.e., $\theta_0$ is totally indifferent), then there are actions $a_0'$ and $a_0''$ such that $\nabla[u(a_0', \theta_0) - u(a_0'', \theta_0] \neq 0$. Without loss, we may assume that $a_0''$ is the action whose utility we normalized to zero for all types at the outset of this proof. Hence, $\nabla f(a_0', \theta_0) \neq 0$. We can then apply Case 1 using $f(a_0', \cdot)$ in place of $f_0(\cdot)$, and one of $f_0(\cdot)$ or $f_1(\cdot)$ in place of $f_1(\cdot)$, because at least one of $f_0(\cdot)$ and $f_1(\cdot)$ is not DSC in a common direction with $f(a_0', \cdot)$.

[29] To elaborate: it must be that for $i = 0, 1$, $f_i$ is neither strictly positive nor strictly negative; otherwise, the two functions would be DSC in a common direction. By continuity, each $f_i$ vanishes on a nonempty set. By Lemma 2, the zero set of each $f_i$ is contained in a hyperplane; and by DSC, the set cannot be a strict (and nonempty) subset of a hyperplane.

all of $\Theta$, or empty. Moreover, the argument in the proof of Proposition 4 shows that $f$ is DSC-preserving.

Say that $g : \Theta \to \mathbb{R}$ is *increasing in direction* $\alpha \in \mathbb{R}^n \setminus \{0\}$ at $\theta$ if for all $\theta'$, it holds that $\alpha \cdot (\theta' - \theta) \geq (\leq) 0 \implies g(\theta') \geq (\leq) g(\theta)$. Note that this is a strengthening of DSC.

**Case 1:** Suppose there are $\theta_0$, $a_0, a_1$ such that $f(a_0, \cdot)$ and $f(a_1, \cdot)$ are not increasing in a common direction at $\theta$.

In this case, $f(a_0, \theta) - f(a_0, \theta_0) f(a^*, \theta)$ and $f(a_1, \theta) - f(a_1, \theta_0) f(a^*, \theta)$ vanish at $\theta_0$, are DSC, but not DSC in a common direction. By a small variant of Lemma 4, $f$ has an affine representative and therefore $u$ has an affine representation.[30]

**Case 2:** Otherwise, for all $\theta$, $a_0, a_1$, the functions $f(a_0, \cdot)$ and $f(a_1, \cdot)$ are increasing in a common direction at $\theta$.

If there is a direction $\alpha \in \mathbb{R}^n \setminus \{0\}$ such that for all $a \in A$ and $\theta \in \Theta$, $f(a, \cdot)$ is increasing in direction $\alpha$ at $\theta$, then $f$ is one-dimensional (see Lemma 1), and hence so is $u$. So suppose there are $a \in A$ and $\theta_0, \theta_1 \in \Theta$ such that $f(a, \cdot)$ is increasing in direction $\alpha \in \mathbb{R}^n \setminus \{0\}$ at $\theta_0$ but not at $\theta_1$. Consider arbitrary $a' \in A$ such that $f(a', \cdot)$ is not constant and note that $f(a', \cdot)$ is increasing in direction $\alpha$ at $\theta_0$. The functions $f_1(\theta) := f(a, \theta) - f(a, \theta_0) f(a^*, \theta)$ and $f_2(\theta) := f(a, \theta) - f(a, \theta_0) f(a^*, \theta)$ vanish at $\theta_0$. Since $\theta_1$ does not lie on the hyperplane in direction $\alpha$ through $\theta_0$ and $f_1(\cdot)$ and $f_2(\cdot)$ are not constant, we have $f_1(\theta_1) \neq 0$ and $f_2(\theta_1) \neq 0$.[31] Hence, the function

$$f_1(\theta) - f_2(\theta) \frac{f_1(\theta_1)}{f_2(\theta_1)}$$

vanishes at $\theta = \theta_1$ and at all $\theta$ such that $(\theta - \theta_0) \cdot \alpha = 0$. Hence, that function is identically zero and therefore there exist $\lambda_0(a'), \lambda_1(a') \in \mathbb{R}$ such that

$$f(a', \theta) = f(a, \theta) \lambda_0(a') + f(a^*, \theta) \lambda_1(a').$$

Since this conclusion also holds if $f(a', \cdot)$ is constant, it holds for all $a' \in A$, contradicting minimal richness. Q.E.D.

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[30] The variant of Lemma 4 is that we replace each $a_i$ ($i = 1, 2, 3$) in the second sentence of the lemma with a finite-support function $\mu_i : A \to \mathbb{R}$ and let $f(\mu_i, \theta) := \sum_{a \in \mathrm{supp}(\mu_i)} f(a, \theta) \mu_i(a)$. The proof of the lemma goes through mutatis mutandis for this variant because $f$ being DSC-preserving implies $f(\mu_i, \theta)$ is DSC.

[31] Indeed, if $f_1(\theta_1) = 0$ then the zero set has nonempty (relative) interior and therefore equals $\Theta$. But then $f_1(\cdot)$ would be increasing in direction $\alpha$ at $\theta_1$, contrary to our supposition. A similar argument shows $f_2(\theta_1) \neq 0$.

## A.7. Connected Choice

The arguments in this subsection hold for any connected topological space $\Theta$, not just a convex $\Theta \subset \mathbb{R}^n$.

**Definition 6.** $u$ has *connected choice* if for all $B \subset A$ and $a \in B$,

$$\left\{\theta : \{a\} = \arg\max_{a' \in B} u(a', \theta)\right\} \text{ is connected.} \tag{13}$$

**Definition 7.** $u$ has *thin indifferences* if for all $B \subset A$ and $a \in B$,

$$\{\theta : u(a, \theta) \geq u(b, \theta)\ \forall b \in B\} \subset \mathrm{cl}\{\theta : u(a, \theta) > u(b, \theta)\ \forall b \in B\}. \tag{14}$$

Thin indifferences can hold without regular indifferences: consider $\Theta = [0, 1]^2$, $A = \{a', a''\}$, and

$$u(a, (\theta_{(1)}, \theta_{(2)})) = \begin{cases} \mathbb{1}\{a = a'\} & \text{if } \theta_{(1)} < 1/2 \text{ or } (\theta_{(1)}, \theta_{(2)}) = (1/2, 0) \\ \mathbb{1}\{a = a''\} & \text{if } \theta_{(1)} > 1/2 \\ 0 & \text{if } \theta_{(1)} = 1/2 \text{ and } \theta_{(2)} > 0. \end{cases}$$

Regular indifferences can also hold without thin indifferences: simply consider a case of total indifference.

**Proposition 7.** *If $u$ has connected choice and thin indifferences, then every locally IC mechanism with finite range is IC.*

The idea of the proof below is as follows. Suppose $m$ is locally IC, but it is not optimal for some type $\theta$ to be truthful. Let $\theta'$ be an optimal report for type $\theta$ and suppose, for simplicity, that it is optimal for type $\theta'$ to be truthful. Let $\Theta_b$ be the types for which $m(\theta')$ is most-preferred in the range of $m$, let $\Theta_1 \subset \Theta_b$ be the types that get $m(\theta')$ under truthtelling, and let $\Theta_2 := \Theta_b \setminus \Theta_1$. Since $\Theta_b$ is connected by assumption, either the closure of $\Theta_1$ intersects $\Theta_2$ or the closure of $\Theta_2$ intersects $\Theta_1$. In either case, local IC is violated, a contradiction.

**Proof of Proposition 7.** Fix a locally IC mechanism $m$ with finite range and suppose there is a type $\theta_0$ for which it is not optimal to be truthful. Let $a := m(\theta_0)$ and let $b \neq a$ be one of type $\theta_0$'s most-preferred alternatives in $m(\Theta)$. Without loss of generality, type $\theta_0$ strictly prefers $b$ to any other alternative.[32]

[32] If not, by thin indifferences, every neighborhood of $\theta_0$ contains a type with $b$ as the uniquely most-

Define $B := \{\theta : m(\theta) = b\}$. Let $A'$ be a maximal subset of $m(\Theta)$ with the property that $a, b \in m\left(\Theta_b^{A'}\right)$, where

$$\Theta_b^{A'} := \left\{\theta : \{b\} = \arg\max_{a' \in A'} u(a', \theta)\right\}.$$

Such a maximal subset exists because $a, b \in m\left(\Theta_b^{\{b\}}\right) = m(\Theta)$ and $m(\Theta)$ is finite. Also note that $\Theta_b^{A'} \not\subset B$ because $\theta_0 \in \Theta_b^{A'}$ and $m(\theta_0) = a$. Since $u$ has connected choice, $\Theta_b^{A'}$ is connected; therefore, $\Theta_b^{A'} \cap B$ and $\Theta_b^{A'} \setminus B$ are not separated.[33] Hence, either (i) there is $\theta \in B \cap \Theta_b^{A'}$ with $m\left(N_\theta \cap \Theta_b^{A'}\right) \not\subset \{b\}$ or (ii) there is $\theta \in \Theta_b^{A'} \setminus B$ with $b \in m\left(N_\theta \cap \Theta_b^{A'}\right)$. In case (i), there is $\theta' \in N_\theta \cap \Theta_b^{A'}$ with $m(\theta') \neq b$; local IC implies $u(m(\theta'), \theta') \geq u(b, \theta')$ and therefore $m(\theta') \notin A'$. In case (ii), there is $\theta' \in N_\theta \cap \Theta_b^{A'}$ with $m(\theta') = b$; local IC implies $u(m(\theta), \theta) \geq u(b, \theta)$ and therefore $m(\theta) \notin A'$. Hence, in either case, there is $\theta_1 \in \Theta_b^{A'}$ and $c \in m(\Theta) \setminus A'$ such that $m(\theta_1) = b$ and $u(b, \theta_1) \geq u(c, \theta_1)$. Since

$$b \in \arg\max_{a' \in A' \cup m\left(N_{\theta_1}\right) \cup \{c\}} u(a', \theta_1)$$

and $u$ has thin indifferences, there is a type $\theta_2 \in N_{\theta_1}$ with

$$\{b\} = \arg\max_{a' \in A' \cup m\left(N_{\theta_1}\right) \cup \{c\}} u(a', \theta_2).$$

Hence, $m(\theta_2) = b$ and the set

$$\Theta_b^{A' \cup \{c\}} := \left\{\theta : \{b\} = \arg\max_{a' \in A' \cup \{c\}} u(a', \theta)\right\}$$

contains $\theta_0$ and $\theta_2$ and therefore satisfies $a, b \in m\left(\Theta_b^{A' \cup \{c\}}\right)$. Hence, $A'$ was not maximal, a contradiction. Q.E.D.

Note that Proposition 7 also holds under the weaker assumption of connected choice in finite choice problems, i.e., the analog of Definition 6 restricted to finite choice sets $B$. The next result shows that this assumption is in fact necessary.

**Proposition 8.** *If $u$ violates connected choice on a finite set,[34] then there is a mechanism that is locally IC but not IC.*

preferred alternative; such a type in $N_{\theta_0}$ does not get $b$ as otherwise local IC would be violated and we can apply our arguments for this type.

[33] Two sets are separated if the closure of each set is disjoint from the other set.

[34] That is, there is a finite set $B \subset A$ and $a \in B$ such that (13) fails.

**Proof.** Suppose there is a finite subset $B \subset A$ and $a \in B$ such that

$$\Theta_a := \left\{\theta : \{a\} = \arg\max_{a' \in B} u(a', \theta)\right\}$$

is not connected. Then $\Theta_a$ can be partitioned into two nonempty sets $\Theta_1$ and $\Theta_2$ that are separated. Note that $|B| \geq 2$, since $\Theta$ is connected. Consider any mechanism $m$ such that $m(\theta) = a$ for all $\theta \in \Theta_1$ and $m(\theta) \in \arg\max_{a' \in B \setminus \{a\}} u(a', \theta)$ for all $\theta \in \Theta \setminus \Theta_1$. Mechanism $m$ is locally IC but not IC. Q.E.D.

We can also show that the assumption of thin indifferences (at least on all finite sets) is necessary in Proposition 7 subject to the following assumption:

$$\text{for all } a \in A \text{ there is } \theta_a \text{ with } \{a\} = \arg\max_{a' \in A} u(a', \theta_a). \tag{15}$$

**Proposition 9.** *Assume (15). If $u$ violates thin indifferences on a finite set,*[35] *then there is a mechanism that is locally IC but not IC.*

**Proof.** Suppose $u$ violates thin indifferences on a finite set $B \subset A$. That is, there are $a \in B$, $\theta_0 \in \Theta$, and a neighborhood of $\theta_0$, call it $N_{\theta_0}$, such that $u(a, \theta_0) \geq u(b, \theta_0)$ for all $b \in B$ and, for all $\theta' \in N_{\theta_0}$, there is $b \in B$ with $u(b, \theta') \geq u(a, \theta')$. By (15), there is $\theta_a$ satisfying $\{a\} = \arg\max_{b \in B} u(b, \theta_a)$. Consider any mechanism $m$ such that $m(\theta_0) = a$ and $m(\theta) \in \arg\max_{b \in B \setminus \{a\}} u(b, \theta)$ for all $\theta \neq \theta_0$. Mechanism $m$ is locally IC but not IC. Q.E.D.

[35] That is, there is a finite set $B \subset A$ and $a \in B$ such that (14) fails.

# Appendix B: Supplementary Appendix

## B.1. On Tightness of Proposition 4's Assumptions

Proposition 4 has three assumptions: $\Theta = \mathbb{R}^n$; $u(a,\theta)$ is differentiable in $\theta$; and no type is totally indifferent (or the weaker version in footnote 20). Example 3 below shows that the first assumption cannot be dropped; Example 4 shows that the latter two cannot be jointly dropped.

**Example 3.** Let $\Theta = \mathbb{R}^2_+$, $X = \{x_0, x_1, x_2\}$, and $A = \Delta X$. Consider expected utility preferences $u : A \times \Theta \to \mathbb{R}$ induced by the vNM utility $\bar{u} : X \times \Theta \to \mathbb{R}$ given by

$$\bar{u}(x,\theta) = \begin{cases} 0 & \text{if } x = x_0 \\ 1 & \text{if } x = x_1 \\ \tan^{-1}\left(\frac{\theta_1}{\theta_2}\right) & \text{if } x = x_2. \end{cases}$$

So $\bar{u}(x_0,\theta)$ and $\bar{u}(x_1,\theta)$ are constant in $\theta$, and, for any $\theta$, there is a hyperplane through $\theta$ such that $\bar{u}(x_2,\theta') \geq \bar{u}(x_2,\theta)$ for $\theta'$ on one side of the hyperplane and $\bar{u}(x_2,\theta') \leq \bar{u}(x_2,\theta)$ for $\theta'$ on the other side. It follows that for any $a \in \Delta X$, $\sum_{i=1}^{3} a(x_i)\bar{u}(x_i,\cdot)$ is DSC, and so there is DSCD over lotteries. However, it can be checked that neither is $u$ one-dimensional nor does it have an affine representation.

**Example 4.** Let $\Theta = \mathbb{R}^2$, $X = \{x_0, x_1, x_2\}$, and $A = \Delta X$. Consider expected utility preferences $u : A \times \Theta \to \mathbb{R}$ induced by the vNM utility $\bar{u} : X \times \Theta \to \mathbb{R}$ given by

$$\bar{u}(x,\theta) = \begin{cases} 0 & \text{if } x = x_0 \\ \theta_1 + \theta_2 & \text{if } x = x_1 \\ \operatorname{sign}(\theta_2)\, e^{\left(\frac{-|\theta_1|}{|\theta_2|}\right)} \sqrt{\theta_1^2 + \theta_2^2} & \text{if } x = x_2. \end{cases}$$

Figure 5 graphs $\bar{u}(x_1,\cdot)$ and $\bar{u}(x_2,\cdot)$. Note that $\bar{u}(x_2,\cdot)$ is not differentiable and type $\theta = 0$ is totally indifferent. It can be verified that there is DSCD over lotteries, but neither is $u$ one-dimensional nor does it have an affine representation.

## B.2. On Grandmont's (1978) Characterization

In this subsection, we elaborate on the connection between Proposition 3 and Grandmont (1978). It is useful to begin with a strengthening of DSCD.

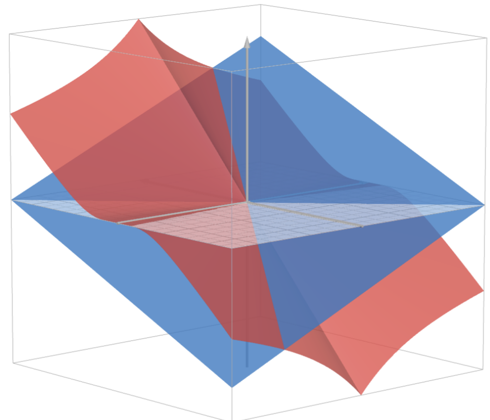

**Figure 5:** The utilities in Example 4, with $\bar{u}(x_1, \theta)$ in blue and $\bar{u}(x_2, \theta)$ in red.

**Definition 8.** A function $f : \Theta \to \mathbb{R}$ is *strictly directionally single crossing (strict DSC)* if it is DSC in some direction $\alpha$ and, in addition, either $f(\cdot) = 0$ or for all $\theta, \theta' \in \Theta$,

$$\Big[(\theta - \theta') \cdot \alpha > 0 \text{ and } f(\theta') = 0\Big] \implies f(\theta) \neq 0.$$

The utility function $u : A \times \Theta \to \mathbb{R}$ has *strict directionally single-crossing differences* (strict DSCD) if for all $a, a' \in A$, the difference $u(a, \theta) - u(a', \theta)$ is strictly DSC.

Strict DSC can be interpreted as requiring that there is a hyperplane such that the sets $\{\theta : f(\theta) < 0\}$ and $\{\theta : f(\theta) > 0\}$ are either empty, full, or intersections of $\Theta$ with half-spaces defined by that hyperplane, and $\{\theta : f(\theta) = 0\}$ is either empty, full, or the intersection of $\Theta$ with that hyperplane. So Figure 2a satisfies strict DSC whereas Figure 2b violates it.

As explained subsequently, the following result follows from Grandmont (1978).

**Proposition 10.** *Assume $\Theta$ is open and that*

$$\forall a, a' \in A, \text{ the set } \{\theta : u(a, \theta) \geq u(a', \theta)\} \text{ is closed in } \Theta. \tag{16}$$

*Then $u$ has convex choice and regular indifferences if and only if it has strict DSCD.*

Hence, under Proposition 10's two assumptions, convex choice and regular indifferences are fully characterized by strict DSCD.[36] By comparison, Proposition 3 required neither assumption and showed that convex choice alone is "almost" characterized by just DSCD.

[36] Strict DSCD implies convex choice and regular indifferences without either assumption. But absent either assumption, strict DSCD can fail despite convex choice and regular indifferences. Here is an example in which $\Theta$ is not open: $\Theta = [0, 1]$, $A = \{a', a''\}$, $u(a', \cdot) = 0$, and $u(a'', \theta) = 0$ if $\theta \in \{0, 1\}$ and $u(a'', \theta) > 0$ otherwise. For an example absent (16), consider the left figure in footnote 16.

To tie Proposition 10 to Grandmont (1978), say that $u$ has *convex weak choice* if for all $a, a' \in A$, the set $\{\theta : u(a,\theta) \geq u(a',\theta)\}$ is convex. In our terminology, Grandmont's condition (H.2) is the conjunction of convex weak choice, convex choice, and regular indifferences.[37] Grandmont's condition (H.1) is our (16). His result can thus be stated as:

**Proposition 11** (Grandmont (1978))**.** *Assume $\Theta$ is open. Then $u$ satisfies* (16) *and has convex weak choice, convex choice, and regular indifferences if and only if for all $a, a' \in A$, either*

1. $\{\theta : u(a',\theta) > u(a,\theta)\} = \Theta$ *or* $\{\theta : u(a',\theta) < u(a,\theta)\} = \Theta$ *or* $\{\theta : u(a',\theta) = u(a,\theta)\} = \Theta$*; or*
2. *there is* $\alpha \in \mathbb{R}^n \setminus 0$ *and* $c \in \mathbb{R}$ *such that* $\alpha \cdot \theta > c$ *if* $u(a,\theta) > u(a',\theta)$*,* $\alpha \cdot \theta < c$ *if* $u(a,\theta) < u(a',\theta)$*, and* $\alpha \cdot \theta = c$ *if* $u(a,\theta) = u(a',\theta)$*.*

The "if" directions of both Proposition 10 and Proposition 11 are straightforward, so let us explain how each proposition's "only if" can be obtained from the other. To go from Proposition 11 to Proposition 10, we can first observe that convex choice and regular indifferences imply convex weak choice when $\Theta$ is open,[38] and then observe that the consequent of Proposition 11 implies strict DSCD. To go from Proposition 10 to Proposition 11, it suffices to observe that given (16), strict DSCD implies the consequent of Proposition 11.[39]

## B.3. On McAfee and McMillan's (1988) Single Crossing

In this subsection, we construct a one-dimensional example in which the utility function satisfies McAfee and McMillan's (1988)'s generalized single crossing (GSC) but does not have convex choice. We construct a mechanism that is locally IC but not IC, as predicted by Proposition 1.

[37] The one-dimensional example in footnote 36 has convex choice and regular indifferences but not convex weak choice. Note that given convex choice and regular indifferences, convex weak choice is equivalent to convex indifference, i.e., for all $a, a'$, the set $\{\theta : u(a,\theta) = u(a',\theta)\}$ is convex.

[38] Suppose that, contrary to convex weak choice, $u(a,\theta_1) \geq u(a',\theta_1)$ and $u(a,\theta_3) \geq u(a',\theta_3)$ but $u(a,\theta_2) < u(a',\theta_2)$ for some $\theta_2 \in \ell(\theta_1,\theta_3)$. If $\Theta$ is open, then there are $\theta_0$ and $\theta_4$ such that $\theta_1, \theta_3 \in \ell(\theta_0,\theta_4)$. If $u(a,\theta) < u(a',\theta)$ for either $\theta \in \{\theta_0,\theta_4\}$, then convex choice fails. If $u(a,\theta) = u(a',\theta)$ for either $\theta \in \{\theta_0,\theta_4\}$, then either regular indifferences or convex choice fails. But if $u(a,\theta) > u(a',\theta)$ for both $\theta \in \{\theta_0,\theta_4\}$, then convex choice fails.

[39] Suppose $u$ has strict DSCD in direction $\alpha \neq 0$ and the sign of $u(a,\cdot) - u(a',\cdot)$ is not constant (otherwise, point 1 of Proposition 11 follows immediately). Then, (16) implies that there is $\theta'$ with $u(a,\theta') = u(a',\theta')$. Consider the hyperplane in the direction of $\alpha$ passing through $\theta'$. Strict DSCD implies that all types on that hyperplane are indifferent between $a$ and $a'$, all types to the left (i.e., types $\theta$ with $\alpha \cdot \theta < \alpha \cdot \theta'$) strictly prefer $a$, and all types to the right strictly prefer $a'$. Point 2 of Proposition 11 follows, with $c = \alpha \cdot \theta'$.

Let $\Theta = [0,1]$, $A = [0,1]$, and $\kappa > 0$.[40] Preferences are given by

$$u(a,\theta) = \begin{cases} -\kappa a - \frac{1}{2}\left(a - \frac{1}{2}\right)^2 \int_0^\theta (2-s)\,\mathrm{d}s & \text{if } a \leq \frac{1}{2} \\ -\kappa a - \frac{1}{2}\left(a - \frac{1}{2}\right)^2 \int_0^\theta (1+s)\,\mathrm{d}s & \text{if } a > \frac{1}{2}. \end{cases}$$

Below, we use subscripts on $u$ to denote partial derivatives.

**Generalized single crossing.** To verify that $u$ satisfies McAfee and McMillan's GSC, we must show that for all $a, \theta, \theta'$ there is $\lambda > 0$ such that

$$u_a(a,\theta) - u_a(a,\theta') = \lambda u_{a\theta}(a,\theta')(\theta - \theta'). \tag{17}$$

Differentiation yields

$$u_a(a,\theta) = \begin{cases} -\kappa - \left(a - \frac{1}{2}\right) \int_0^\theta (2-s)\,\mathrm{d}s & \text{if } a \leq \frac{1}{2} \\ -\kappa - \left(a - \frac{1}{2}\right) \int_0^\theta (1+s)\,\mathrm{d}s & \text{if } a > \frac{1}{2}, \end{cases}$$

and

$$u_{a,\theta}(a\theta) = \begin{cases} -\left(a - \frac{1}{2}\right)(2-\theta) & \text{if } a \leq \frac{1}{2} \\ -\left(a - \frac{1}{2}\right)(1+\theta) & \text{if } a > \frac{1}{2}. \end{cases}$$

If $u_{a\theta}(a,\theta) > 0$ then $a < \frac{1}{2}$, hence

$$u_a(a,\theta) - u_a(a,\theta') = -\left(a - \frac{1}{2}\right) \int_{\theta'}^{\theta} (2-s)\,\mathrm{d}s$$

is strictly positive (negative) if and only if $\theta' < (>)\theta$. Similarly, if $u_{a\theta}(a,\theta) < 0$ then $a > \frac{1}{2}$, hence

$$u_a(a,\theta) - u_a(a,\theta') = -\left(a - \frac{1}{2}\right) \int_{\theta'}^{\theta} (1+s)\,\mathrm{d}s$$

is strictly positive (negative) if and only if $\theta' < (>)\theta$. Lastly, if $u_{a\theta}(a,\theta) = 0$ then $a = \frac{1}{2}$ and $u_a(a,\theta) - u_a(a,\theta') = 0$. Therefore, for all $a, \theta, \theta'$ there is $\lambda > 0$ satisfying Equation 17.

[40] McAfee and McMillan (1988) allow for transfers. We implicitly set transfers to zero to simplify.

**No convex choice and the insufficiency of local IC.** Let $\kappa > 0$ be small and compare $a = 0$ with $a = 1$:

$$u(0,\theta) = -\frac{1}{2}\left(\frac{1}{2}\right)^2 \int_0^\theta (2-s)\,\mathrm{d}s,$$
$$u(1,\theta) = -\kappa - \frac{1}{2}\left(\frac{1}{2}\right)^2 \int_0^\theta (1+s)\,\mathrm{d}s.$$

Hence,

$$\begin{aligned} u(1,\theta) - u(0,0) &= \int_0^\theta [(2-s)-(1+s)]\,\mathrm{d}s - 8\kappa \\ &= \int_0^\theta (1-2s)\,\mathrm{d}s - 8\kappa \\ &= \theta - \theta^2 - 8\kappa, \end{aligned}$$

with roots

$$\underline{\theta} = \frac{1-\sqrt{1-32\kappa}}{2} \text{ and } \overline{\theta} = \frac{1+\sqrt{1-32\kappa}}{2}.$$

These two types are indifferent between the actions $a = 0$ and $a = 1$, types in $\left(\underline{\theta}, \overline{\theta}\right)$ strictly prefer $a = 1$, and types outside $\left[\underline{\theta}, \overline{\theta}\right]$ strictly prefer $a = 0$. So convex choice—and hence also DSCD—fails. Moreover, the following mechanism is locally IC but not IC:

$$m(\theta) = \begin{cases} 0 \text{ if } \theta \leq \underline{\theta} \\ 1 \text{ otherwise.} \end{cases}$$