An Efficient Dynamic Auction for General Economies with Indivisibilities *

Moritz Drexl Andreas Kleiner†

27th October 2015

Abstract

We study general economies with indivisibilities in which agents can be producers and/or consumers of multiple units of heterogeneous goods and have substitutes preferences. We derive a simple tâtonnement process from a steepest descent algorithm and use this process to construct an incentive compatible and efficient dynamic auction. This allows us to reinterpret the price adjustment process discovered by Ausubel (2006) as a steepest descent algorithm. Our results provide an incentive compatible and efficient dynamic auction for the substitutes and complements setting introduced by Sun and Yang (2006) and the trading network economy of Hatfield, Kominers, Nichifor, Ostrovsky and Westkamp (2013). We also provide a variant of this auction that uses only singleton demand reports.

JEL Classification: C02; C61; C62; D44; D47; D51

Keywords: Tâtonnement, Ascending Auction, Competitive Equilibrium, Trading Networks, Gross Substitutes, Gross Substitutes and Complements, Discrete Convex Analysis

1 Introduction

The idea of a tâtonnement process that tentatively adjusts prices according to current supply and demand was formulated by Walras (1874), and has since then been used to derive ascending iterative auctions and methods for finding market-clearing prices. Walras also realized that a form of substitutability of demand was needed for such a process to work. For divisible goods, a formal description of a tâtonnement process and convergence results were first given by Samuelson (1947) and Arrow and Hurwicz (1958). In settings with indivisible goods, the first formal tâtonnement process was an algorithm described by Kelso and Crawford (1982) for the allocation of workers to firms. They realized that in these settings it is sufficient for convergence that the agents' demand satisfies the gross substitutes condition. In subsequent papers, different algorithms were developed for the

*We are indebted to Werner Hildenbrand, Benny Moldovanu, David Parkes and Zaifu Yang for helpful comments.

†Drexl, Kleiner: Bonn Graduate School of Economics, University of Bonn, Kaiserstr. 1, 53113 Bonn, Germany; drexl@uni-bonn.de, akleiner@uni-bonn.de

same condition (Gul and Stacchetti 1999, Ausubel 2006, Milgrom and Strulovici 2009). As observed recently by Fujishige and Yang (2003), the demand function of an agent satisfies gross substitution precisely if the valuation function that describes the agent's preferences is $M^\natural$-concave (that is, it belongs to a class of well-behaved discrete concave functions). In this paper, we make use of this connection to design and analyze discrete tâtonnement processes and efficient dynamic auctions.

To be more precise, we look at economies where every agent has quasi-linear preferences over bundles of goods. The economy may initially be endowed with arbitrary quantities of the goods in order to model auction environments, and we assume that individuals are endowed with a sufficient amount of money so that they will never face budget constraints and are able to buy any bundle of goods as they wish. We allow for models where agents have preferences over negative amounts of goods so that agents may be buyers and/or sellers, as well as models that do not satisfy the assumption of free disposal. We use a suitable extension of gross substitute valuation functions to multiple units of goods, namely $M^\natural$-concave valuation functions. The paper proposes a tâtonnement process based on a steepest descent algorithm. This price adjustment process is then used to construct an incentive compatible and efficient dynamic auction for this general economy. Based on results from Discrete Convex Analysis, we can give simple and intuitive proofs for the convergence and incentive properties of the dynamic auction. We also construct a variant of this auction that uses only singleton demand reports. All auctions previously proposed for discrete settings (starting with the seminal work by Demange, Gale and Sotomayor 1986) instead required that agents report their entire demand sets. That is, if a bidder is indifferent between multiple bundles at the current price he must report all bundles. Milgrom and Strulovici (2009) emphasize that this feature makes the proposed auctions "different from any auction process in current use".[1] Using insights about the structure of demand, we construct an exact auction that uses singleton demand reports and that is incentive compatible.

Applying our results to Ausubel's (2006) model, we can reinterpret his dynamic auction as a steepest descent algorithm. Arguing that an $M^\natural$-convex function is the appropriate extension of a gross substitutes valuation function to multiple units of goods, our price adjustment process has the additional advantage of generating linear prices for multiple units of goods (as in Milgrom and Strulovici 2009). Moreover, using this extension we can accommodate agents that act as sellers and/or buyers. Allowing for sellers is important for many real-world applications: for example, recent spectrum auctions had the goal to reallocate spectrum rights from current owners to agents that value them more. Our generalization provides an efficient dynamic auction for such settings.

Since we allow for models where agents can be buyers and/or sellers as well as models that do not satisfy the assumption of free disposal, our model also accommodates the substitutes and complements framework by Sun and Yang (2006): This setting, with two classes of goods such that items in the same class are substitutes and items in different classes are complements to each other, can be obtained by just "mirroring" all valuation

[1] Demange et al. (1986) and Milgrom and Strulovici (2009) propose approximate auctions to circumvent this problem. While the approximate auction of Demange et al. (1986) converges to a price that is close to a competitive equilibrium price vector if all bidders report their demand truthfully, bidders have not necessarily an incentive to do so. The approximate auction of Milgrom and Strulovici (2009) does in general not stop close to a competitive equilibrium price vector.

functions along the axes of the goods in one class. Using this observation, our algorithm generalizes the double-track adjustment process proposed by Sun and Yang (2009) to multiple units of goods (see also Shioura and Yang 2013). Moreover, applying our dynamic auction to this setting yields an incentive compatible and efficient dynamic auction for the substitutes and complements framework. Our model also contains the trading network economy of Hatfield et al. (2013) as a special case and their full substitutes condition is equivalent to the convexity assumption we use in this paper. We therefore provide a price adjustment process and an efficient dynamic auction for their trading network economy.

The interpretation of tâtonnement processes as discrete steepest descent algorithms also has the advantage that statements about the algorithmic complexity of the process can be made. Further, efficient scaling techniques can be applied which yield adjustment processes that only need to state a strongly polynomial number of ask prices, albeit at the cost of monotone convergence.

$M^\natural$-convexity turns out to be the right notion of convexity in the discrete setting because $M^\natural$-convex functions exhibit several properties that are important for the establishment of a discrete price adjustment process. First, the class of $M^\natural$-convex functions is closed under aggregation, which implies that aggregate demand shares the same properties as every agent's demand. Since therefore, aggregate demand is convex,[2] every bundle of goods is demanded at some price, which means that a competitive equilibrium exists. The notion of $M^\natural$-convexity is, however, too strong to be necessary for the existence of competitive equilibria. We refer to Danilov, Koshevoy and Murota (2001) and Baldwin and Klemperer (2013) for a complete characterization of the classes of valuation functions that guarantee existence of an equilibrium. $M^\natural$-convex functions provide, additionally, the appropriate combinatorial properties such as convexity and submodularity of the aggregate indirect utility function which are key for the (monotone) convergence of discrete steepest descent algorithms.

Another key property that is implied by $M^\natural$-convexity is that the optimal descent direction of the indirect utility function at a given price is entirely determined by the aggregate demand correspondence. Therefore, even though indirect utility cannot be elicited directly, the steepest descent algorithm is economically suitable because it can proceed through best response information from the agents. The algorithm can therefore be applied to construct incentive compatible and efficient dynamic auctions. This point also highlights the link between the optimization results that were independently obtained in economics and mathematics: Even though in economic settings the auction process has to be described in terms of the bidders' demand sets, and descent methods in optimization theory are applied to convex functions directly, both approaches describe the same algorithms.

Steepest descent methods for the design of iterative auctions are complemented by primal-dual and linear programming algorithms (Demange et al. 1986, Gul and Stacchetti 1999, Parkes and Ungar 2000, deVries, Schummer and Vohra 2007) and related to algorithms for finding stable outcomes in matching models (Gale and Shapley 1962, Ostrovski 2008).

[2]The appropriate notion is convex-extensibility, see Definition 3 below.

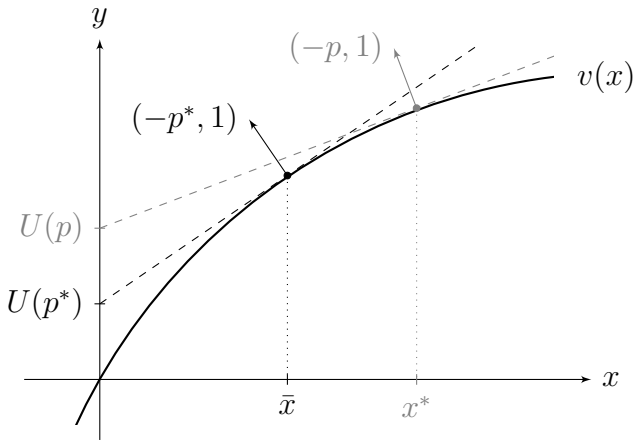


Figure 1: Concavity and conjugacy.

Illustration: Convexity and Tâtonnement

The role of convexity and conjugacy in the establishment of a price adjustment process can best be described visually by means of a valuation function with a continuous domain. In order to keep things simple, we assume that the aggregate valuation function v of a group of agents over quantities x of a single good is given (see Figure 1). Aggregate utility is quasi-linear, and therefore a competitive equilibrium price for endowment $\bar{x}$ is a price p^* such that $\bar{x}$ maximizes the expression $v(x) - p^*x$, i.e., a price such that the agents demand a quantity of precisely $\bar{x}$. Alternatively, the maximization problem of the agents can be written as maximizing the linear function $(-p, 1)^T(x, y)$ over points (x, y) such that $y \leq v(x)$. Since v is concave, it is then clear that for an equilibrium price p^*, the vector $(-p^*, 1)$ will be perpendicular to the tangent of the convex set $V = \{(x, y) \mid y \leq v(x)\}$ at the point $(\bar{x}, v(\bar{x}))$. It is then also clear that concavity of v guarantees that an equilibrium price exists for every endowment.

In order to derive a process that converges to the equilibrium price we look at the indirect utility at price p, which is defined as $U(p) = \max_x v(x) - px$. If x^* attains the maximum, we can write $v(x^*) = U(p) + px^*$ and therefore $U(p)$ is the intercept of the tangent of the set V at the point $(x^*, v(x^*))$. Also, since v is concave, we can see from Figure 1 that the value $v(x)$ can be recovered from the intercepts $U(p)$ via $v(x) = \min_p U(p) + px$. This is called conjugacy and the functions v and U are said to be conjugate to each other. Since U is defined as the maximum over a family of linear functions, it will be convex.

If p^* is an equilibrium price for endowment $\bar{x}$, then $v(\bar{x}) - p^*\bar{x} = U(p^*)$ and therefore, because of conjugacy, p^* attains $\min_p U(p) + p\bar{x}$. Equilibrium prices can therefore be found by computing a minimizer of the function $h(p) = U(p) + p\bar{x}$. Since U is convex, h is convex as well, and there are well-developed algorithms for minimizing convex functions. Some examples are steepest descent and gradient methods.

This paper demonstrates that these considerations can be applied to valuation functions with discrete domains as well by using the theory of Discrete Convex Analysis. The paper is structured as follows: Section 2 introduces the notation and basic model. Different restrictions on preferences that are equivalent to gross substitutes are discussed in Section 3. Then, Section 4 covers the existence and properties of competitive equilibria.

Section 5 presents the price adjustment process and Section 6 constructs an incentive compatible dynamic auction. Finally, in Section 7, we apply our results and conclude in Section 8. A brief introduction to Discrete Convex Analysis is provided in the Appendix.

2 Economy

There is a set of agents N and a set of *heterogeneous goods* G. The economy is endowed with positive or negative[3] quantities of goods $\bar{x} \in \mathbb{Z}^G$. Typically, in an auction setting, $\bar{x}$ will be the vector of goods for sale and in a pure exchange economy, we will typically have $\bar{x} = 0$. We denote by X the set of feasible aggregate consumption bundles and assume that this set is finite.[4] Each agent $i \in N$ has a *valuation function* $v_i : X \to \mathbb{R} \cup \{-\infty\}$ over bundles of goods (where multiple units of a good are allowed).[5] We extend the domain of the valuation function without loss of generality to $\mathbb{Z}^G$ by setting $v_i(x) = -\infty$ for all $x \in \mathbb{Z}^G \setminus X$. Agents have *quasi-linear utilities*. In particular, if linear prices $p \in \mathbb{R}^G$ are given, agent i derives utility $u_i(x, p) = v_i(x) - \langle p, x \rangle$ from bundle $x \in \mathbb{Z}^G$. Agent i's *indirect utility function* is defined as

$$U_i(p) = \max_{x \in X} v_i(x) - \langle p, x \rangle$$

for all $p \in \mathbb{R}^G$. The *demand correspondence* $D_i(p)$ is the set of maximizers in the expression above.

Definition 1. *A* competitive equilibrium *is an allocation of goods $x^i \in \mathbb{Z}^G$, $i \in N$ with $\sum_{i \in N} x^i = \bar{x}$, together with a price vector $p^* \in \mathbb{R}^G$ such that $x^i \in D_i(p^*)$ for every agent $i \in N$.*

3 Preferences and Discrete Concavity

In settings where only one unit of every good is available and where there is no production, the gross substitutes property has turned out to be sufficient for the existence of competitive equilibria and the design of price adjustment processes. However, when multiple units of a good are available, this condition is too weak. In this section we provide a sensible generalization of the gross substitutes condition to our setting. The following is a definition of the gross substitutes property, naively adapted to multiple units of goods:

Definition 2. *The valuation v_i satisfies* weak substitutes (wGS)[6] *if for every pair of prices $p \leq p'$, and every $x \in D_i(p)$, there exists some $x' \in D_i(p')$ such that $x_j \leq x'_j$ for every j for which $p_j = p'_j$.*

[3]Negative endowments are important for the application to the double-track adjustment process by Sun and Yang (2009), see Section 7.

[4]This is a technical restriction that is satisfied in all auction settings and holds in environments with producers if it is infeasible to produce infinite quantities of a good.

[5]We will assume that v_i is integer-valued from Section 5 on.

[6]Historically, gross substitution is a condition on the demand correspondence of an agent. However, through the specific definition of D_i above, it can be defined in terms of the valuation function v_i. The same applies to the other definitions given in this section.

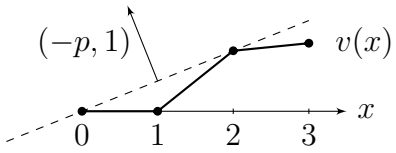


Figure 2: Non-existence of competitive equilibria: There is no price such that the agent demands a quantity of one.

To see why this condition is too weak for the existence of a competitive equilibrium, note that it is vacuous for the case of only one good. See the example in Figure 2 for an illustration of non-existence of an equilibrium (one good, one agent). The figure suggests that a form of concavity is required for the existence of a competitive equilibrium. Our requirement on valuation functions (Assumption 1 below) implies that the function is concave in the following sense. Also see Figures 4a and 4b for an illustration.

Definition 3. *A valuation function v_i is* concave-extensible *if it coincides with its concave closure $\bar{v}_i : \mathbb{R}^G \to R \cup \{-\infty\}$, given by*

$$\bar{v}_i(x) = \inf_{p \in \mathbb{R}^G, \alpha \in \mathbb{R}} \{\langle p, x\rangle + \alpha \mid \langle p, x\rangle + \alpha \geq v_i(z) \ \forall z \in \mathbb{Z}^G\},$$

on the set of integer vectors, i.e., if $v_i(x) = \bar{v}_i(x)$ for all $x \in \mathbb{Z}^G$.

When restricted to the unit cube $\{0, 1\}^G$, there are several properties that are equivalent to gross substitutes: For instance, a valuation function satisfies the gross substitutes condition if and only if it satisfies the step-wise gross substitutes condition (Danilov, Koshevoy and Lang 2003):

Definition 4. *Valuation function v_i satisfies* step-wise gross substitutes (SWGS) *if for any $p \in \mathbb{R}^G, x \in D_i(p)$ and $j \in G$, we either have*

(i) $x \in D_i(p + \delta \mathbf{1}_j)$ for all $\delta \geq 0$ or

(ii) there is some $\delta \geq 0$ and $x' \in D_i(p + \delta \mathbf{1}_j)$ with $x'_j = x_j - 1$ and $x'_{-j} \geq x_{-j}$.[7]

The second property which is on the unit cube equivalent to gross substitutes is the single-improvement property (Gul and Stacchetti 1999):

Definition 5. *The valuation function v_i satisfies the* single-improvement property*, if for any price p, and bundles x, y such that $u_i(y, p) > u_i(x, p)$, there exists a bundle x' such that $u_i(x', p) > u_i(x, p)$ and $x' = x + \mathbf{1}_j - \mathbf{1}_k$ with $j \in \text{supp}^+(y - x) \cup \{0\}$ and $k \in \text{supp}^-(y - x) \cup \{0\}$.*[8]

Murota and Tamura (2002) show that for general valuation domains, a valuation function satisfies the single-improvement property if and only if it is concave-extensible and satisfies step-wise gross substitutes. Therefore, these properties are suitable generalizations of gross substitutes to multiple units of goods.

[7] We denote by $\mathbf{1}_S \in \mathbb{R}^E$ the *characteristic vector* that equals 1 for $e \in S$ and 0 otherwise. We write $\mathbf{1}_e = \mathbf{1}_{\{e\}}$ and $\mathbf{1}_0 = (0, ..., 0)$.

[8] We define the *negative* and *positive support* of a vector $x \in \mathbb{R}^G$ as $\text{supp}^-(x) = \{g \in G | x(g) < 0\}$ and $\text{supp}^+(x) = \{g \in G | x(g) > 0\}$.

Assumption 1. *For every agent i, the valuation function v_i is concave-extensible and satisfies the step-wise gross-substitutes property. Equivalently, v_i satisfies the single-improvement property.*

The interpretation that an agent with a valuation function satisfying this requirement views the different goods as substitutes remains valid also for multiple units. First, on the unit cube, valuations satisfying Assumption 1 are precisely those that satisfy the gross substitutes property as defined by Kelso and Crawford (1982). Second, for non-negative goods vectors, these valuations are precisely the strong substitutes valuations as defined in Milgrom and Strulovici (2009). A valuation satisfies *strong substitutes* if, when every unit of every good is treated as a separate good, the valuation satisfies gross substitutes with respect to them.

It is also known (Fujishige and Yang 2003, Murota and Tamura 2002) that a function satisfies Assumption 1 if and only if it is $M^\natural$-concave. These functions form a class of well-behaved concave functions that play an important role in Discrete Convex Analysis and exhibit combinatorial properties that allow the design of an iterative tâtonnement process. An introduction to $M^\natural$-concave functions is given in the appendix. For the reader unfamiliar with Discrete Convex Analysis, the appendix also provides a brief exposition of the main results about discrete convex functions that are used in the next two sections. A completely self-contained treatment of the topic can be found in Murota (2003).

4 Competitive Equilibrium

This section is concerned with the existence and properties of competitive equilibria. As we show below, existence of competitive equilibria is a property of the aggregate valuation function: If it is concave in the sense that the superdifferential is always non-empty, then an equilibrium exists for every endowment (also see Danilov et al. 2001, Baldwin and Klemperer 2013). This is always fulfilled for integrally convex functions and $M^\natural$-concave functions in particular. For the case of free disposal, we also show that equilibrium prices will always be non-negative.

The definition of the indirect utility function $U_i(p)$ and the *concave conjugate* $v_i^\circ(p) = \inf_{x\in X}\{\langle p, x\rangle - v_i(x)\}$ for $p \in \mathbb{R}^G$ implies $U_i = -v_i^\circ$.[9] Let v_N denote the aggregate valuation function of all agents in N, i.e.,

$$v_N(x) = \sup_{\{x^i\}_{i\in N}} \left\{ \sum_{i\in N} v_i(x^i) \,\middle|\, \sum_{i\in N} x^i = x \right\}.$$

This aggregate valuation function is just the convolution (which is defined in (A.6) in the appendix) of all the agents' valuation functions. Then, writing U_N for the indirect utility function of the whole group of agents, we get

$$U_N = -v_N^\circ = -\sum_{i\in N} v_i^\circ = \sum_{i\in N} U_i \tag{1}$$

by (A.7). The aggregate demand of all agents $D_N(p)$ is defined as the set of demands attaining $U_N(p)$. This allows us to rephrase the definition of competitive equilibrium.

[9] We define the *discrete concave conjugate* $v_i^{\circ_\mathbb{Z}}$ as the restriction of v_i° to $\mathbb{Z}^G$.

Proposition 1. *Price vector p^* is a competitive equilibrium price vector if and only if $\bar{x} \in D_N(p^*)$.*

Proof. First, let $x^i \in D_i(p^*)$ and $\sum_{i \in N} x^i = \bar{x}$. Then $U_i(p^*) = v_i(x^i) - \langle p^*, x^i \rangle \geq v_i(x^i) - \langle p^*, y^i \rangle$ for all allocations $\{y^i\}_{i \in N}$, $\sum_{i \in N} y^i = \bar{x}$. Summing up these inequalities implies $\sum_{i \in N} v_i(x^i) \geq \sum_{i \in N} v_i(y^i)$ (cf. First Welfare Theorem) and therefore $v_N(\bar{x}) = \sum_{i \in N} v_i(x^i)$.

Using (1), we have

$$U_N(p^*) = \sum_{i \in N} [v_i(x^i) - \langle p^*, x^i \rangle] = v_N(\bar{x}) - \langle p^*, \bar{x} \rangle,$$

which implies $\bar{x} \in D_N(p^*)$.

Conversely, let $\bar{x} \in D_N(p^*)$. We get from the definition of v_N an allocation $\bar{x} = \sum_{i \in N} x^i$ with $v_N(\bar{x}) = \sum_{i \in N} v_i(x^i)$. Then we get

$$\sum_{i \in N} U_i(p^*) = U_N(p^*) = v_N(\bar{x}) - \langle p^*, \bar{x} \rangle = \sum_{i \in N} [v_i(x^i) - \langle p^*, x^i \rangle],$$

where we in turn use (1), $\bar{x} \in D_N(p^*)$, and the definition of v_N. Since $U_i(p^*) \geq v_i(x^i) - \langle p^*, x^i \rangle$ for all $i \in N$, these inequalities need to hold with equality and therefore $x^i \in D_i(p^*)$ for all $i \in N$. □

The proof also implies that $D_N(p)$ is equal to the Minkowski sum $\sum_{i \in N} D_i(p)$, which is defined as $A + B = \{a + b \mid a \in A \text{ and } b \in B\}$. The proposition indicates that the existence of competitive equilibria is a property of the aggregate valuation function v_N. If it is concave in the sense that for every endowment $\bar{x}$ the superdifferential at $\bar{x}$ is non-empty, then a competitive equilibrium is guaranteed to exist (see the appendix for a definition of the superdifferential). Since, as the convolution of $M^\natural$-concave functions, v_N is $M^\natural$-concave, this is the case in our setting.

Theorem 1. *In the economy defined above, if the valuation function of every agent satisfies Assumption 1 and $\bar{x} \in \operatorname{dom} v_N$, a competitive equilibrium exists.*[10]

Proof. Since v_i is $M^\natural$-concave for all $i \in N$, the aggregate valuation function v_N is also $M^\natural$-concave by (A.7). Also note that for $p \in \mathbb{Z}^G$ and $x \in \operatorname{dom} v_N$, we have $x \in D_N(p) \Leftrightarrow p \in \partial' v_N(x)$. By Theorem A.1 (i) in the Appendix, for $M^\natural$-concave functions $\partial' v_N(\bar{x})$ is non-empty and therefore there exists p^* such that $\bar{x} \in D_N(p^*)$, that is, a competitive equilibrium exists. □

Remark 1. *Theorem A.1 also implies that the set of competitive equilibrium prices is an $L^\natural$-convex set. In particular this means that it is a lattice.*

Although we do not need to impose free disposal for our results, the following proposition establishes the intuitive fact that, whenever there is free disposal, prices are non-negative in equilibrium.

Proposition 2. *Assume that v_i is non-decreasing for all $i \in N$, that is $v_i(x) \leq v_i(y)$ for $x \leq y$. Then for every competitive equilibrium price p^* we have $p^* \geq 0$.*

[10] We denote the effective domain by $\operatorname{dom} v_i = \{z \in \mathbb{Z}^E \mid v_i(z) \neq -\infty\}$

Proof. We first show that if v_1 and v_2 are non-decreasing, then $v_1 \,\square\, v_2$ is non-decreasing. Then by induction it follows that v_N is non-deceasing. So let $x \leq y$ and $x^1 + x^2 = x$ such that $(v_1 \,\square\, v_2)(x) = v_1(x^1) + v_2(x^2)$. Since v_2 is non-decreasing, we have

$$v_1(x^1) + v_2(x^2) \leq v_1(x^1) + v_2(x^2 + [y - x]).$$

Then $(v_1 \,\square\, v_2)(x) \leq (v_1 \,\square\, v_2)(y)$ follows because $y = x^1 + x^2 + [y - x]$.

Now take $j \in G$ and define $y = \bar{x} + \mathbf{1}_j$. Since $p^* \in \partial' v_N(\bar{x})$, we have $v_N(\bar{x}) - \langle p^*, \bar{x} \rangle \geq v_N(\bar{y}) - \langle p^*, y \rangle$. Therefore

$$0 \leq v_N(y) - v_N(\bar{x}) \leq \langle p^*, y - \bar{x} \rangle = p^*_j,$$

which completes the proof. □

5 Tâtonnement

From now on we assume that valuations are integer-valued, i. e. $v_i : \mathbb{Z}^G \to \mathbb{Z} \cup \{-\infty\}$. Conjugacy between the aggregate valuation and indirect utility function helps us to easily see that the set of competitive equilibrium prices coincides with the set of minimizers of a Liapunov function (Ausubel 2006, Milgrom and Strulovici 2009). Since this function will be $L^\natural$-convex, steepest descent algorithms can be used for the computation of competitive equilibrium prices. In this section, we present such an algorithm and analyze its convergence properties. We also make use of the fact that the subdifferential of an $L^\natural$-convex function determines its slope to show how one can compute the descent direction via the demand sets. Finally we provide an alternative algorithm that uses as input singleton demand reports. We will use these results to construct an incentive compatible dynamic auction in the next section.

Proposition 3. *Price vector p^* supports endowment $\bar{x}$ in competitive equilibrium if and only if it minimizes the function $h(p) = \langle p, \bar{x} \rangle + U_N(p)$. Moreover, h is minimized by an integer price vector.*

Proof. By Proposition 1, a price vector p^* supports $\bar{x}$ in competitive equilibrium if and only if

$$v_N(\bar{x}) - \langle p^*, \bar{x} \rangle = U_N(p^*) = -v_N^\circ(p^*).$$

By conjugacy we know that the concave extension $\bar{v}_N$ coincides with its biconjugate, i. e. $\bar{v}_N = v_N^{\circ\circ}$ and therefore $v_N(\bar{x}) = \inf_p \{\langle p, \bar{x} \rangle - v_N^\circ(p)\}$. Hence, necessity follows since p^* attains the infimum and therefore minimizes h. Conversely, if p^* minimizes h, it attains the infimum and therefore constitutes an equilibrium price vector.

Moreover, by discrete conjugacy (Theorem A.3 in the appendix) we know that v_N coincides with its discrete biconjugate, i. e. $v_N = v_N^{\circ_\mathbb{Z} \circ_\mathbb{Z}}$, and hence $v_N(\bar{x}) = \inf_{p \in \mathbb{Z}^G} \{\langle p, \bar{x} \rangle - v_N^{\circ_\mathbb{Z}}(p)\}$. Since h is integer-valued on $\mathbb{Z}^G$, it attains its infimum on $\mathbb{Z}^G$. □

For general valuation functions, a price that minimizes h is often referred to as a *quasi-equilibrium* (Milgrom and Strulovici 2009). Theorem A.3 in the Appendix implies that U_i and hence h are integral polyhedral $L^\natural$-convex. In the following we will present a steepest descent algorithm that minimizes the function h and therefore constitutes a price adjustment process for the economy considered. The correctness of the algorithm follows from the following optimality criterion for the minimization of $L^\natural$-convex functions:

Proposition 4. *For an $L^\natural$-convex function h we have that $h(p) \leq h(q)$ for all $q \in \mathbb{Z}^G$ if and only if $h(p) \leq h(p \pm \mathbf{1}_S)$ for all $S \subseteq G$.*

Proof. We only need to show sufficiency. With regard to Proposition A.1 in the Appendix we need to show that $h(q) \geq h(p)$ for all $q \in \mathbb{Z}^G$ with $||q - p||_\infty \leq 1$. Every such q can be written as $q = p + \mathbf{1}_X - \mathbf{1}_Y$ for suitable disjoint $X, Y \subseteq G$. Then submodularity of h and local optimality of p imply

$$h(p) + h(p + \mathbf{1}_X - \mathbf{1}_Y) \geq h(p + \mathbf{1}_X) + h(p - \mathbf{1}_Y) \geq 2h(p),$$

which completes the proof. □

It is therefore straightforward to use the following algorithm for minimizing h: Start with an arbitrary price vector p and search for a subset of goods S and $\varepsilon \in \{-1, 1\}$ such that $h(p + \varepsilon \mathbf{1}_S) - h(p)$ is minimal. If no subset S and ε can be found such that this difference is negative, p is a competitive equilibrium. Otherwise update the price to $p + \varepsilon \mathbf{1}_S$ and iterate. Since we have integer valuations, h decreases by at least 1 in every step and therefore (since a competitive equilibrium exists) the algorithm converges after finitely many steps.[11] It is summarized in Algorithm 1.

Algorithm 1 Steepest Descent

1. Pick an arbitrary price vector $p \in \operatorname{dom} h = \mathbb{Z}^G$.
2. **while** there exists $\varepsilon \in \{-1, 1\}$ and $S \subseteq G$ with $h(p + \varepsilon \mathbf{1}_S) < h(p)$ **do**
3. Choose S, ε such that $h(p + \varepsilon \mathbf{1}_S) - h(p)$ is minimized.
4. Set $p := p + \varepsilon \mathbf{1}_S$.
5. **end while**
6. p is a competitive equilibrium price vector.

Monotone Convergence

Although we have already seen that the algorithm converges globally, it is often desirable to have an algorithm that converges monotonically (i.e., for iterative combinatorial auctions). The following tie-breaking rule implies that if the algorithm starts with a price that is below every equilibrium price and always sets $\varepsilon = +1$, then it monotonically converges to the lowest equilibrium price:

$$\text{Choose the (unique) minimal minimizer } S \text{ of } h(p + \mathbf{1}_S) - h(p). \tag{2}$$

A unique minimal minimizer exists because h is submodular.

Let p^* be the lowest equilibrium price (such a price exists since the set of competitive equilibrium prices is a lattice). Convergence follows if the modified algorithm never stops strictly below p^* and always stays below p^*. The former property is a consequence of the integral convexity of h; the latter property follows from the tie-breaking rule and the submodularity of h.

Lemma 1. *If the modified algorithm stops at p, then $p \geq p^*$.*

[11] In fact, this argument implies the convergence of *any* descent algorithm.

Proof. Assume that the algorithm stops at p but $p < p^*$. Then there exists $\lambda \in (0,1)$ such that $p' = \lambda p + (1-\lambda)p^*$ has the property that $||p' - p||_\infty < 1$. The integral neighborhood $N(p')$ defined in (A.4) consists of vectors $p + \mathbf{1}_X$ for subsets $X \subseteq G$. Since the algorithm stopped at p, $h(p + \mathbf{1}_X) \geq h(p)$ for all these subsets. Hence, by the definition of the local convex extension (A.5), $\tilde{h}(p') \geq \tilde{h}(p)$. But this is a contradiction because $\tilde{h}(p') < \tilde{h}(p) = h(p)$ due to the convexity of $\tilde{h}$, which holds by integral convexity of h. □

Lemma 2. *If $p' = p + \mathbf{1}_S$ is chosen by the modified algorithm at $p \leq p^*$ then $p' \leq p^*$.*

Proof. Submodularity of h implies

$$h(p^*) + h(p') \geq h(p^* \vee p') + h(p^* \wedge p').$$

Since p^* is an equilibrium price vector, we have $h(p^*) \leq h(p^* \vee p')$, and hence $h(p') \geq h(p^* \wedge p')$. Now assume that there is a $j \in G$ with $p'_j > p^*_j$. Then $p^* \wedge p' = p + \mathbf{1}_T$ for $T \subseteq S \setminus \{j\}$, which contradicts the minimality of S prescribed by the tie-breaking rule (2). □

For instance, if free-disposal can be assumed, then every competitive equilibrium price is always non-negative and hence the monotone tâtonnement process can always be started with a price of $p = 0$ and converges to the lowest equilibrium price.

We get the following summarizing result:

Theorem 2. *If the valuation function of every agent $i \in N$ satisfies Assumption 1, then for every endowment $\bar{x} \in \mathbb{Z}^G$ and every initial price vector $p \in \mathbb{Z}^G$, Algorithm 1 converges to a competitive equilibrium price vector.*

Further, if the starting price vector is lower than every equilibrium price, then the monotonic version of Algorithm 1 converges to the lowest competitive equilibrium price.

Eliciting Descent Directions

In practice, for instance when using the tâtonnement process as an iterative auction, it is impractical to elicit the values of the indirect utility functions $U_i(p)$ and therefore impossible to evaluate $h(p + \varepsilon\mathbf{1}_S)$. However, when every agent reports his demand correspondence $D_i(p)$ at the current price p, it is possible to compute the difference $h(p + \varepsilon\mathbf{1}_S) - h(p)$ for every ε and S (Ausubel 2006). The intuitive reason for this is that for an $L^\natural$-convex function h, the difference between the function values at $p + \varepsilon\mathbf{1}_S$ and p can be constructed from the subdifferential $\partial h(p)$ of h at the point p. The following lemma demonstrates that this subdifferential corresponds to the set of excess supply vectors.

Lemma 3. *For any $p \in \mathbb{Z}^G$ we have $x \in D_N(p) \;\Leftrightarrow\; \bar{x} - x \in \partial h(p)$.*

Proof. By Proposition 1, $x \in D_N(p)$ is equivalent to p being a competitive equilibrium price vector for endowment x, which by Proposition 3 is equivalent to

$$U_N(p) + \langle p, x\rangle \leq U_N(q) + \langle q, x\rangle$$

for all q. This is just the definition of $-x \in \partial U_N(p)$, which by the definition of h is equivalent to $\bar{x} - x \in \partial h(p)$. □

Proof. This is a consequence of the conjugacy between v_N and U_N. First note that by the definition of h, we have $\bar{x} - x \in \partial h(p) \quad \Leftrightarrow \quad -x \in \partial U_N(p)$. Next, $x \in D_N(p)$ is equivalent to $U_N(p) = v_N(x) - \langle p, x\rangle$. Since $v_N(x) = \inf_q\{U_N(q) + \langle q, x\rangle\}$ by conjugacy, this is equivalent to

$$U_N(p) + \langle p, x\rangle \leq U_N(q) + \langle q, x\rangle \quad \forall q \in \mathbb{Z}^N.$$

This is in turn just the definition of $-x \in \partial U_N(p)$, which completes the proof. □

Since h is an $L^\natural$-convex function, the difference between $h(p + \varepsilon \mathbf{1}_S)$ and $h(p)$ can be computed via the support function of its subgradient $\partial h(p)$ at p, evaluated in the direction of $\varepsilon \mathbf{1}_S$:

Lemma 4. *Let g be an integer-valued polyhedral $L^\natural$-convex function, $p \in \operatorname{dom} g$ and $S \subseteq G$, $\varepsilon \in \{-1, 1\}$. Then*

$$g(p + \varepsilon \mathbf{1}_S) - g(p) = \max_{y \in \partial g(p)} \langle y, \varepsilon \mathbf{1}_S\rangle.$$

For a proof see, for example, Proposition 7.44 in Murota (2003). The difference $h(p + \varepsilon \mathbf{1}_S) - h(p)$ can now be computed as follows. By Lemma 3,

$$\max_{y \in \partial h(p)} \langle y, \varepsilon \mathbf{1}_S\rangle = \max_{x \in D_N(p)} \langle \bar{x} - x, \varepsilon \mathbf{1}_S\rangle$$

and therefore, we have established the following proposition:

Proposition 5. *Assume that x^* solves the optimization problem*

$$\min_{x \in D_N(p)} \langle x, \varepsilon \mathbf{1}_S\rangle. \tag{3}$$

Then, by Lemma 4,

$$h(p + \varepsilon \mathbf{1}_S) - h(p) = \langle \bar{x} - x^*, \varepsilon \mathbf{1}_S\rangle.$$

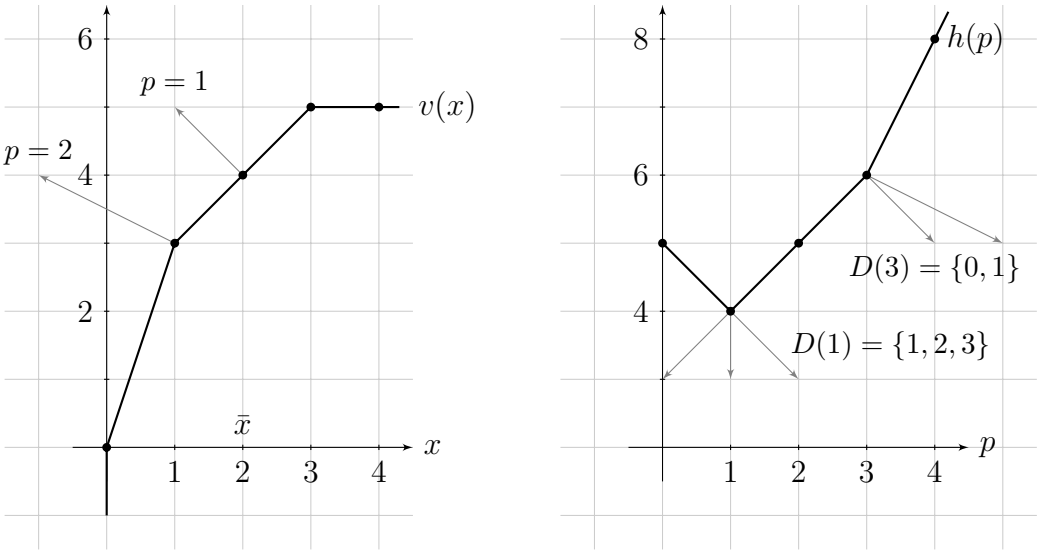
The optimization problem (3) can be decomposed by choosing $x^{i*} \in \arg\min_{x^i \in D_i(p)} \langle x^i, \varepsilon \mathbf{1}_S\rangle$ for every agent separately and then setting $x^* = \sum_{i \in N} x^{i*}$, since the objective function is linear and $D_N(p)$ is the Minkowski sum of the sets $D_i(p)$.[12] Also, since the sets $D_i(p)$ are $M^\natural$-convex, the greedy algorithm provides a way to maximize a linear objective function over $D_i(p)$ efficiently (Dress and Wenzel 1990).

While the method described in Proposition 5 can be used to evaluate $h(p+\varepsilon \mathbf{1}_S) - h(p)$, it remains to find $\varepsilon \in \{-1, 1\}$ and S such that this term is minimized. Since h is $L^\natural$-convex and in particular submodular, $h(p+\varepsilon \mathbf{1}_S) - h(p)$ is submodular in (ε, S), and efficient algorithms for minimizing submodular set functions can be used (Schrijver 2000).

Remark 2. *If one does not insist on a consecutive price trajectory but instead allows the ask price to jump around freely, the price adjustment algorithm can be scaled. The resulting algorithm then finds a competitive equilibrium in strongly polynomial time (see Murota 2003).*

The following example illustrates how the descent direction can be derived from demand reports for the case of one agent.

[12] Ausubel (2006) gives a different proof of a version of Proposition 5 which makes use of the single-improvement property that v_N satisfies since it is $M^\natural$-concave. Instead, we use the $L^\natural$-convexity of U_N and h.



(a) Valuation function used in Example 1.

(b) The function h corresponding to the valuation function on the left.

Figure 3: Illustration of the relation between demand correspondence and subdifferential in Example 1.

Example 1. *Assume that there is one agent with the valuation function v depicted in Figure 3a and that the economy is endowed with $\bar{x} = 2$ units of one good. The corresponding function $h(p) = 2p + U(p)$ is shown in Figure 3b. Let the price adjustment process start with $p = 3$. At this price, the agent will demand quantities of either 0 or 1. In accordance with Lemma 3, the subdifferential of h at $p = 3$ is $\{1, 2\}$. Slopes in the direction of -1 and 1 are given by $1 \cdot (-1) = -1$ and $2 \cdot 1 = 2$, respectively, and therefore the price should be adjusted downwards. At $p = 2$, the agent only demands a quantity of 1 and the price should be lowered further. At $p = 1$, the agent demands $D(1) = \{1, 2, 3\}$, and the subdifferential of h at $p = 1$ is $\{-1, 0, 1\}$. Since the slope in the direction of -1 and 1 is $(-1) \cdot (-1) = 1$ and $1 \cdot 1 = 1$, respectively, we know that $p = 1$ is a minimum of the function h and that we have found an equilibrium.*

Singleton-based tâtonnement

The previous development uses the fact that descent directions of a convex function can be derived from knowledge of its complete subdifferential correspondence. This fact has been used in previous auction formats but relies on the knowledge of the complete aggregate demand correspondence (see also Gul and Stacchetti 2000, Ausubel 2006). Milgrom and Strulovici (2009) however emphasize that in all practical auctions that are currently used, bidders only report one desired bundle at a given price and not their entire demand sets. According to their argument, none of the proposed auctions for discrete objects can be implemented under current practices.

We now suggest a price adjustment process that uses only singleton demand reports, thereby corresponding to common auction designs, but that is nonetheless guaranteed to stop at a competitive equilibrium. Let $d_N(p)$ denote an arbitrary selection from $D_N(p)$.

Algorithm 2 Singleton-based algorithm

Pick an arbitrary price vector $p \in \mathbb{Z}^G$.
while there exists $\varepsilon \in \{-1, 1\}$ and $S \subseteq G$ with $\langle \bar{x} - d_N(p), \varepsilon \mathbf{1}_S \rangle < 0$ such that (ε, S) has not been chosen in this iteration **do**
  Choose such a pair S, ε.
  if $\langle \bar{x} - d_N(p + \frac{\varepsilon}{2}\mathbf{1}_S), \frac{\varepsilon}{2}\mathbf{1}_S \rangle < 0$ **then**
    Set $p := p + \varepsilon \mathbf{1}_S$.
  else
    Set $p := p$.
  end if
end while
p is a competitive equilibrium price vector.

If agents report only singleton demands, their reported demands might not be market-clearing even at a competitive equilibrium price. The above algorithm therefore has to use additional information to determine how to adjust prices and when to stop. This information is obtained by eliciting demand at additional prices and using the structure of the indirect utilities implied by convexity.

Theorem 3. *If the valuation function of every agent $i \in N$ satisfies Assumption 1, then for every endowment $\bar{x} \in \mathbb{Z}^G$ and every initial price vector $p \in \mathbb{Z}^G$, Algorithm 2 converges to a competitive equilibrium price vector.*

Proof. Note that for any $\varepsilon \in \{-1, 1\}$ and $S \subseteq G$, $\max_{y \in \partial h(p)} \langle y, \varepsilon \mathbf{1}_S \rangle = \langle x, \varepsilon \mathbf{1}_S \rangle$ for $x \in \partial h(p + \frac{\varepsilon}{2}\mathbf{1}_S)$: Convexity of h implies that $\max_{y \in \partial h(p)} \langle y, \varepsilon \mathbf{1}_S \rangle \leq \langle x, \varepsilon \mathbf{1}_S \rangle$ and a strict inequality would contradict Lemma 4.

If the price is adjusted from p to $p + \varepsilon \mathbf{1}_S$ it holds that $\langle \bar{x} - d_N(p + \frac{\varepsilon}{2}\mathbf{1}_S), \varepsilon \mathbf{1}_S \rangle < 0$. By Lemma 3, $\bar{x} - d_N(p + \frac{\varepsilon}{2}\mathbf{1}_S) \in \partial h(p + \frac{\varepsilon}{2}\mathbf{1}_S)$, which implies

$$h(p + \varepsilon \mathbf{1}_S) - h(p) = \langle \bar{x} - d_N(p + \frac{\varepsilon}{2}\mathbf{1}_S), \varepsilon \mathbf{1}_S \rangle < 0.$$

Hence, the value of h decreases by at least one for every price change and the algorithm terminates after finitely many iterations.

If the algorithm stops at p, it holds that for every $\varepsilon \in \{-1, 1\}$ and $S \subseteq G$,

$$h(p + \varepsilon \mathbf{1}_S) - h(p) \geq 0.$$

By Proposition 4, p is a global minimizer of h and hence p supports endowment $\bar{x}$ in competitive equilibrium by Proposition 3. □

6 Incentive Compatibility

In this section we show how the above algorithms can be used to construct incentive compatible dynamic auctions.

The auction is modeled as a dynamic game in discrete time with mandatory participation. Each period t the auctioneer announces a price $p(t)$. Each player i then

reports her set $x^i(t) \subset \mathbb{Z}^G$ of optimal consumption bundles. Let H_t^i denote the history of play before period t that is observable to player i. This could for example be $H_t^i = H_t = \{p(s) \text{ and } x^j(s) : \ 0 \leq s < t \text{ and } j \in N\}$. A strategy σ^i for player i is then a sequence of mappings $\sigma_t^i : \mathbb{Z}^G \times H_t^i \to 2^{\mathbb{Z}^G}$ that select for each period, each price, and each history the subset of bundles that are demanded. Denote by Σ_i the set of strategies.

Given price $p(t)$ and the reports in period t, the auctioneer runs one iteration of Algorithm 1 to determine the price $p(t+1)$ for the next period and computes

$$x^{i*}(t) \in \arg\min_{x \in x^i(t)} \langle x, p(t+1) - p(t) \rangle.$$

The game then moves to the next period. If the algorithm returns a competitive equilibrium in period T, the game ends.[13] Each player receives an allocation corresponding to this competitive equilibrium price vector and makes a payment

$$a_i(T) = \sum_{t=0}^{T-1} \left[\sum_{j \neq i} \langle x^{j*}(t), p(t+1) - p(t) \rangle \right] - \langle p(T), \bar{x} - x^{i*}(T) \rangle.$$

We say that bidder i bids *sincerely* relative to utility function v_i if, at every time t, her bid $x^i(t)$ equals $D_i(p(t))$.

Theorem 4. *Sincere bidding by every bidder is an ex post perfect equilibrium of the auction game.*

Proof. Fix sincere reports for all agents except agent i. Given this fixed behavior by all other agents, any strategy $\sigma^i \in \Sigma_i$ induces a list of reports $\hat{x}^i(t)$, which yields payoff $v_i(\hat{x}^{i*}(T)) - a_i(T)$.

Since v_j is $M^\natural$-concave, $U_j = -v_j^\circ$ is $L^\natural$-convex. Lemma 4 therefore implies that $U_j(p(t+1)) - U_j(p(t)) = -\langle x^{j*}(t), p(t+1) - p(t) \rangle$ for all $j \neq i$. We therefore get

$$\begin{aligned}
& v_i(\hat{x}^{i*}(T)) - a_i(T) \\
=& v_i(\hat{x}^{i*}(T)) + \sum_{t=0}^{T-1} \left[\sum_{j \neq i} U_j(p(t+1)) - U_j(p(t)) \right] + \langle p(T), \bar{x} - \hat{x}^{i*}(T) \rangle \\
=& v_i(\hat{x}^{i*}(T)) + \sum_{j \neq i} \left[U_j(p(T)) - U_j(p(0)) \right] + \langle p(T), \bar{x} - \hat{x}^{i*}(T) \rangle \\
=& v_i(\hat{x}^{i*}(T)) + \sum_{j \neq i} \left[v_j(x^{j*}(T)) - U_j(p(0)) \right] \\
\leq& \sum_{j \in N} v_j(x^{CE}) - \sum_{j \neq i} U_j(p(0)),
\end{aligned}$$

where the last equality uses that $\bar{x} = \hat{x}^{i*}(T) + \sum_{j \neq i} x^{j*}(T)$ if the algorithm stops in period T and x^{CE} denotes a competitive equilibrium allocation for the true preferences. The final inequality follows from the First Welfare Theorem.

[13] If the algorithm never stops, we assign each agent an payoff of $-\infty$. By adjusting the payments we could dispense with this penalty and induce agents to participate voluntarily. However, since our setting includes producers (and in particular the Myerson-Satterthwaite setting), this will in general create a budget deficit for the designer.

Reporting truthfully achieves equality in the above expression because the algorithm then chooses a competitive equilibrium with respect to the true preferences and therefore sincere bidding is an ex post perfect equilibrium. □

An analogous construction can be used for an auction that relies on Algorithm 2. Thereby we can construct an dynamic auction that requires bidders only to report one of their preferred bundles.

7 Applications

In this section we demonstrate how different models from the literature fit in our framework and how our results can be applied to them.

Ausubel's Auction for Heterogeneous Goods

In a seminal paper, Ausubel (2006) developed a price adjustment process for auction settings with indivisible goods where agents' preferences satisfy the gross substitutes property. The auction proceeds through the minimization of a Lyapunov function and the analysis in this paper is heavily inspired by this. Conversely, Ausubel's (2006) auction is an important special case of our adjustment process:

Corollary 1. *Assume that agents have valuation functions over the unit cube* $\{0,1\}^G$ *and that the economy is endowed with one unit of each good* ($\bar{x} = \mathbf{1}_G$)*. Then the tâtonnement process outlined in Section 5 describes the discrete price adjustment process presented in Ausubel (2006).*

Thus, our model generalizes Ausubel's model in that it works for preferences over arbitrary positive and/or negative quantities of every good, as well as any initial endowment. While positive quantities other than one can be simulated in Ausubel's framework by modeling every unit as a separate good, the auction then results in non-linear prices. In contrast, our algorithm generates linear prices for arbitrary quantities.

Milgrom and Strulovici (2009) also generalize Ausubel to multiple units of goods and introduce the strong substitutes condition, which, for positive quantities, is equivalent to Assumption 1. Therefore, our work also generalizes Milgrom and Strulovici (2009) by allowing for negative quantities and therefore for the possibility to model producers.

Our framework can also be applied to the set of preferences that are used in the Product-Mix Auction introduced by Klemperer (2010), since these preferences are a special case of gross substitutes.

Gross Substitutes and Complements

The double-track adjustment process presented in Sun and Yang (2009) is a special case of the price adjustment process outlined above. We start by recalling the gross substitutes and complements condition. In the model introduced by Sun and Yang (2006), the set of goods is partitioned into two sets $G = G_1 \sqcup G_2$.

Definition 6. *A valuation $v_i : \{0,1\}^G \to \mathbb{R}$ satisfies the* weak/ordinary gross substitutes and complements (GSC) *condition, if, given some price vector $p \in \mathbb{R}^G$, some good*[14] *$j \in G_a$, and $\delta > 0$, the following holds: For every $x \in D_i(p)$ there exists $x' \in D_i(p + \delta \mathbf{1}_j)$ such that for all $k \neq j$, we have $x_k \leq x'_k$ if $k \in G_a$ and $x_k \geq x'_k$ if $k \in G_b$.*

We show that every valuation that satisfies the GSC condition can be transformed into a valuation that satisfies the GS condition by reversing the sign of every good in G_2. Assume that the goods are ordered such that the goods in G_1 come before the goods in G_2. Then the transformation can be described by applying the matrix

$$M = \begin{pmatrix} I_{|G_1|} & 0 \\ 0 & -I_{|G_2|} \end{pmatrix},$$

where $I_{|G_a|}$ is the identity matrix of dimension $|G_a|$. Using this transformation we can define the transformed valuation function $M^* v_i$ through $M^* v_i(x) = v_i(Mx)$. The transformed indirect utility $M^* U_i$ and demand correspondence $M^* D_i$ are defined using the transformed valuation function.

Lemma 5. *We have $x \in D_i(p)$ if and only if $M^{-1}x \in M^* D_i(Mp)$.*

Proof. By definition, we have $x \in D_i(p)$ if and only if $v_i(x) - \langle p, x \rangle \geq v_i(x') - \langle p, x' \rangle$ for all $x' \in \operatorname{dom} v_i$. By substituting $x = My$ and $x' = My'$, this is equivalent to

$$\begin{aligned} & v_i(My) - \langle p, My \rangle \geq v_i(My') - \langle p, My' \rangle \\ \Leftrightarrow \quad & M^* v_i(y) - \langle Mp, y \rangle \geq M^* v_i(y') - \langle Mp, y' \rangle \quad \forall y \in \operatorname{dom} M^* v_i, \end{aligned}$$

which in turn means that $y = M^{-1}x \in M^* D_i(Mp)$. □

Proposition 6. *Let $v_i : \{0,1\}^G \to \mathbb{R}$. Then v_i satisfies GSC if and only if $M^* v_i$ satisfies wGS (i.e., is $M^\natural$-concave).*

Proof. First note that wGS is equivalent to a version where the price of only one good is increased. We show equivalence to this modified definition.

Assume that v_i satisfies GSC. Let $p \in \mathbb{R}^G$, $\delta > 0$ and $j \in G_1$. Define $p' = p + \delta \mathbf{1}_j$ and let $x \in M^* D_i(p)$. We need to find $x' \in M^* D_i(p')$ such that for $k \neq j$, $x_k \leq x'_k$. By Lemma 5 we know that $y = Mx \in D_i(Mp)$. Also, since $j \in G_1$, $M(p + \delta \mathbf{1}_j) = Mp + \delta \mathbf{1}_j$. Since v_i satisfies the GSC condition, we know that there exists $y' \in D_i(Mp + \delta \mathbf{1}_j)$ such that for all $k \neq j$, we have $y_k \leq y'_k$ if $k \in G_1$ and $y_k \geq y'_k$ if $k \in G_2$.

We claim that $x' = M^{-1}y'$ satisfies the requirements. First, by Lemma 5, $x' \in M^* D_i(p + \delta \mathbf{1}_j)$. Now take some good $k \neq j$. If $k \in G_1$ then $x_k = y_k$ and $x'_k = y'_k$ and therefore $x_k \leq x'_k$. If $k \in G_2$, then $x_k = -y_k$ and $x'_k = -y'_k$ and therefore $x_k = -y_k \leq -y'_k = x'_k$.

The argument is similar for the case where $j \in G_2$ and also sufficiency can be shown analogously. □

Proposition 6 motivates the following definition of generalized gross substitutes and complements for multiple units of goods (cf. Baldwin and Klemperer 2013, Shioura and Yang 2013):

[14] In this definition, a and b are set to 1 and 2, or 2 and 1, respectively.

Definition 7. *Valuation v_i satisfies the* generalized gross substitutes and complements (GGSC) *condition, if M^*v_i is $M^\natural$-concave.*

With this definition, existence of competitive equilibria follows immediately: For a set of valuation functions $\{v_i\}_{i\in N}$ that satisfy GGSC and endowment $\bar{x}$, consider the modified economy $\{M^*v_i\}_{i\in N}$ with endowment $M\bar{x}$. Since $\{M^*v_i\}_{i\in N}$ are $M^\natural$-concave, there exists a competitive equilibrium price vector p^*, that is, $M\bar{x} \in M^*D_N(p^*)$. Then, by Lemma 5, $\bar{x} \in D_N(Mp^*)$, so Mp^* is a competitive equilibrium price vector for the original economy.

We note that the application of our results via the described transformation above requires us to be able to deal with non free disposal valuations. Specifically, if a valuation v_i satisfies free disposal then M^*v_i has "anti free disposal" for goods in G_2. It follows as in Proposition 2 that the price p_j^* for $j \in G_2$ is non-positive, and therefore Mp_j^* is non-negative.

The above transformation also allows us to describe the double-track price adjustment process by Sun and Yang (2006) in terms of the algorithm from Section 5: The algorithm is run on the modified economy $\{M^*v_i\}_{i\in N}$ and $M\bar{x}$ (call it *internal representation*). If we transform this algorithm back to the original economy (call it *external representation*), we get the price adjustment process described in Sun and Yang (2009). In particular,

(i) if the current internal price is Mp, the price p is presented to the agents. If the internal price for some good in G_2 increases, then the external price decreases and vice versa.

(ii) if an agent indicates that he demands bundle x, then $M^{-1}x$ is used for the internal calculation of the next price.

(iii) if the monotone convergence algorithm is used internally, the starting price has to be set such that it is below every competitive equilibrium price. In the original economy, this means that the price for goods in G_1 has to be set to the lowest and the price for goods in G_2 to the highest possible level. Then, since the algorithm converges monotonically in the internal representation, this means that the real price for goods in G_1 increases whereas the real price for goods in G_2 decreases.

(iv) since the set of (internal) equilibrium prices is a lattice, the set of transformed equilibrium prices forms a "generalized lattice" as defined by Sun and Yang (2009).

Moreover, the result in Section 6 provides an incentive compatible dynamic auction. Hence, we can formulate the following corollary of Theorems 1, 2, and 4.

Corollary 2. *Assume that agents have valuation functions over the unit cube $\{0,1\}^G$ and that these valuation functions satisfy GSC. Further, assume that $\bar{x} = \mathbf{1}_G$. Then, a competitive equilibrium exists and the procedure outlined above describes the double-track adjustment process presented in Sun and Yang (2009).*

Thus, the results in this paper generalize Sun and Yang (2006) as well as Sun and Yang (2009) in that they work for preferences over arbitrary positive and/or negative quantities of every good, as well as for any initial endowment of the economy, if the valuation functions satisfy the GGSC condition.

Trading Networks

The trading networks economy introduced by Hatfield et al. (2013) also fits into our model. We first describe the network economy and then show how the valuation functions in our paper relate to the valuation functions as they are defined in Hatfield et al. (2013).

In the model, there is a set of agents N and a set of trades Ω which can be interpreted as goods. The agents and trades form a graph, where the nodes are the agents and each trade is a directed edge. If trade $\omega = (i, i') \in \Omega$ points from agent i to agent i' then we say that agent i is the seller and agent i' is the buyer in this trade. Let Ω_i be the trades that are adjacent to agent i. Every agent i has a valuation function $v_i : \{-1, 0, 1\}^{\Omega_i} \to \mathbb{R}$ over subsets of adjacent trades. We model agent i being a buyer in trade $\omega \in \Omega_i$ by requiring that for $x \in \{-1, 0, 1\}^{\Omega_i}$, $v_i(x) = -\infty$ if $x_\omega = -1$. Similarly, if agent i is a seller in trade ω we require $v_i(x) = -\infty$ whenever $x_\omega = 1$. The interpretation is that if agent i demands vector x with $x_\omega = -1$ then he wants to be engaged in trade ω where he is the seller and similarly, if $x_\omega = 1$ then he wants to be engaged in trade ω where he is the buyer.

We can embed this economy in our model by extending the valuation functions v_i to $\{-1, 0, 1\}^{\Omega}$ as follows: Set $v_i(x) = -\infty$ if $x_\omega \neq 0$ for some $\omega \notin \Omega_i$. Otherwise, if $x_\omega = 0$ for all $\omega \notin \Omega_i$, just copy the valuation of the vector x restricted to Ω_i. A competitive equilibrium in the network economy is a competitive equilibrium for the endowment $\bar{x} = 0$. Then we know that, whenever all valuation functions v_i satisfy Assumption 1, there exists a competitive equilibrium and a convergent price adjustment process.

We therefore get the following corollary regarding the model by Hatfield et al. (2013):

Corollary 3. *In the model defined above, if the valuation function of every agent satisfies Assumption 1, a competitive equilibrium exists. Further, Algorithm 1 can be used to find competitive equilibrium prices for any initial price vector p. Section 6 provides an incentive compatible dynamic auction for the model defined above.*

In the following we explain how Assumption 1 is equivalent to the full substitutes condition defined in Hatfield et al. (2013). In their paper, valuation functions, utility functions, demand, and indirect utility are defined slightly differently as follows: Every agent i has a valuation function $\tilde{v}_i : \{0, 1\}^{\Omega_i} \to \mathbb{R}$ that can be embedded into $\{0, 1\}^{\Omega}$ as described above. Let $\Omega_{i\to}$ be the trades adjacent to agent i in which he is a seller and let $\Omega_{\to i}$ be the trades adjacent to him in which he is a buyer, respectively. Then the interpretation is that if agent i demands bundle x and $x_\omega = 1$ then agent i wants to be engaged in trade ω. Given price vector $p \in \mathbb{R}^{\Omega}$, an agent's quasi-linear utility is defined as

$$\tilde{u}_i(x, p) = \tilde{v}_i(x) + \sum_{\omega \in \Omega_{i\to} : x_\omega = 1} p_\omega - \sum_{\omega \in \Omega_{\to i} : x_\omega = 1} p_\omega.$$

Indirect utility $\tilde{U}_i$ and demand correspondence $\tilde{D}_i$ are then defined as in Section 2, but using $\tilde{u}_i$.

Hatfield et al. (2013) assume full substitutability which is defined as follows:[15]

Definition 8. *A valuation function v_i satisfies* full substitutability (FS) *if for every two price vectors $p \leq p'$ the following holds: For every $x \in \tilde{D}_i(p)$ there exists $x' \in \tilde{D}_i(p')$ such that whenever $p_\omega = p'_\omega$ for some ω, then $x_\omega \leq x'_\omega$ if $\omega \in \Omega_{\to i}$ and $x_\omega \geq x'_\omega$ if $\omega \in \Omega_{i\to}$.*

[15] This formulation is similar and equivalent to "indicator language full substitutability."

We now show that full substitutability and gross substitutability are equivalent. Fix some agent i. We introduce the following transformation of a vector $x \in \{-1, 0, 1\}^{\Omega}$. As in the last subsection, assume that the trades are ordered such that trades $\omega \in \Omega_{i\to}$ come first. Then we apply the following matrix:

$$M = \begin{pmatrix} -I_{|\Omega_{i\to}|} & 0 \\ 0 & I_{|\Omega \setminus \Omega_{i\to}|} \end{pmatrix}$$

From the interpretation of the valuation functions v_i and $\tilde{v}_i$ we see that for them to represent the same preferences over trades, $v_i(Mx) = \tilde{v}_i(x)$ has to hold for all $x \in \{0, 1\}^{\Omega}$.

Lemma 6. *For transformed bundles we have $Mx \in D_i(p) \Leftrightarrow x \in \tilde{D}_i(p)$.*

Proof. This follows from

$$\sum_{\omega \in \Omega_{i\to} : x_\omega = 1} p_\omega - \sum_{\omega \in \Omega_{\to i} : x_\omega = 1} p_\omega = -\langle p, Mx \rangle$$

and $v_i(Mx) = \tilde{v}_i(x)$. □

Proposition 7. *A valuation $\tilde{v}_i$ satisfies FS if and only if the corresponding valuation function v_i satisfies Assumption 1.*

Proof. On the unit-cube, Assumption 1 is equivalent to the ordinary (weak) gross substitutes condition. After applying the translation $v_i'(x) = v_i(x - \mathbf{1}_{\Omega_{i\to}})$, $\operatorname{dom} v_i'$ is the unit-cube. Since the gross substitutes condition is translation-invariant, it is therefore enough to show that $\tilde{v}_i$ satisfies FS if and only if v_i satisfies weak GS (Definition 2).

In order to prove necessity assume that $\tilde{v}_i$ satisfies FS. Take price vectors $p \leq p'$ and $x \in D_i(p)$. By Lemma 6 we know that $Mx \in \tilde{D}_i(p)$. Since $\tilde{v}_i$ satisfies FS, we know that there exists $Mx' \in \tilde{D}_i(p')$ such that whenever $p_\omega = p'_\omega$ for some ω, then $Mx_\omega \leq Mx'_\omega$ if $\omega \in \Omega_{\to i}$ and $Mx_\omega \geq Mx'_\omega$ if $\omega \in \Omega_{i\to}$. Hence, for ω with $p_\omega = p'_\omega$ we have $x_\omega \leq x'_\omega$. Furthermore, $x' \in D_i(p')$ and therefore x' satisfies all requirements in Definition 2.

Sufficiency is proved similarly. □

Combining the transformation M with the translation v' in the proof above yields the same transformation as that which is used in the proof for the existence of competitive equilibria in Hatfield et al. (2013). However, the translation is not needed in our model since our framework can deal with negative amounts of goods (that is, producers).

We can also use our framework to extend the model in this subsection to multiple units of goods in each trade. Further, the transformation in the last subsection on the gross substitutes and complements condition can be applied to the trading network model to get two sets of trades Ω^1 and Ω^2 where trades in the same sets substitute each other but trades in different sets are complements (see also Drexl 2013).

8 Discussion

In this paper we have used the theory of Discrete Convex Analysis to unify and generalize the literature on tâtonnement for economies with indivisibilities. The interpretation of the auction procedure proposed by Ausubel (2006) as a steepest descent algorithm of

certain discrete convex functions yields simple and intuitive proofs of the convergence properties of the generalized adjustment process. Applying the results demonstrates that all the literature on discrete tâtonnement harnesses the notion of gross substitutability, which is equivalent to $M^\natural$-convexity. The theory of Discrete Convex Analysis confirms that $M^\natural$-convexity is essential for many properties of discrete convex functions.

Since the existence of market-clearing equilibria can be guaranteed for classes of valuation functions that are much more general than the valuations we consider (see Baldwin and Klemperer 2013), one of the big open questions is whether a price adjustment process can be designed that converges for every instance of valuation functions where equilibria are guaranteed to exist. While the indirect utility function in these cases is still convex, it does not exhibit all the combinatorial properties used in the present paper. Still, it might very well be possible to prove convergence of a suitably designed descent or gradient algorithm that minimizes the aggregate indirect utility function.

Appendix: Discrete Convex Analysis

In this section we review the main definitions and results from Discrete Convex Analysis. For a complete and self-contained treatment of the topic we refer the reader to Murota (2003).

A convex function f with a convex domain in $\mathbb{R}^G$ has several attractive properties. First, local optimality implies global optimality. This yields many efficient optimization methods for convex functions. Second, by the supporting hyperplane theorem, the subdifferential of a convex function is non-empty everywhere, and the function can be recovered from the set of subdifferentials. This implies conjugacy and duality results for the convex conjugate (or Legendre-Fenchel transform) of f. The theory of Discrete Convex Analysis identifies classes of convex functions defined on a subset of the discrete lattice $\mathbb{Z}^G$ for which discrete analogues of the above properties hold. These play an important role in establishing the results in this paper.

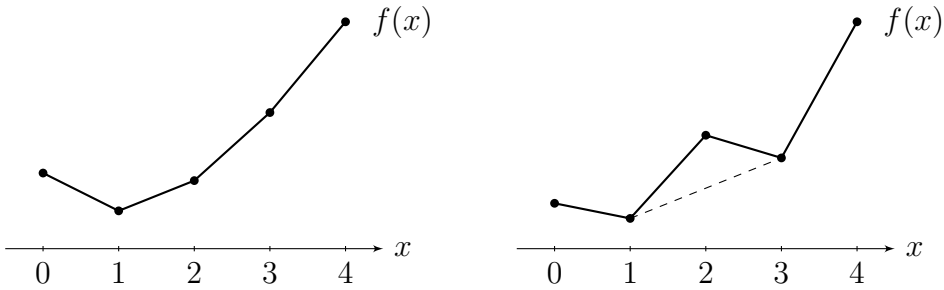
The first important property of functions $f : \mathbb{Z}^G \to \mathbb{R}$, which will be shared by the two subclasses of $M^\natural$- and $L^\natural$-convex functions, is integral convexity. It is defined in terms of suitable convex extensions of f to a real-valued domain. Define the *convex closure* $\bar{f}$ of f as

$$\bar{f}(x) = \sup_{p \in \mathbb{R}^G, \alpha \in \mathbb{R}} \{\langle p, x\rangle + \alpha \mid \langle p, x\rangle + \alpha \leq f(y) \ \forall y \in \mathbb{Z}^G\}.$$

This is equivalent to taking the convex hull of the epigraph of f. If the convex closure coincides with f on the set of integer vectors, i.e., if $f(x) = \bar{f}(x)$ for all $x \in \mathbb{Z}^G$, f is called *convex-extensible*, see Definition 3. We can relax the requirement in the above definition to obtain a local version of the convex extension: The *integral neighborhood* $N(x)$ of $x \in \mathbb{R}^G$ is defined as

$$N(x) = \{y \in \mathbb{Z}^G \ : \ ||y - x||_\infty < 1\}. \tag{A.4}$$

If we only impose the inequality $\langle p, x\rangle + \alpha \leq f(y)$ for points y in the integral neighborhood



(a) An integrally convex function with local convex extension.

(b) A function that is not integrally convex, together with its local convex extension and convex closure.

Figure 4: Local convex extension, convex closure and integral convexity.

of x, we get the *local convex extension* $\tilde{f}$ of f, which is defined as

$$\begin{aligned}\tilde{f}(x) &= \sup_{p\in\mathbb{R}^G,\alpha\in\mathbb{R}} \{\langle p,x\rangle + \alpha \mid \langle p,x\rangle + \alpha \leq f(y)\ \forall y \in N(x)\} \\ &= \inf\left\{ \sum_{y\in N(x)} \lambda_y f(y) \;\middle|\; \sum_{y\in N(x)} \lambda_y y = x, \sum_{y\in N(x)} \lambda_y = 1, \lambda_y \geq 0 \right\}. \end{aligned} \tag{A.5}$$

Here, equality of the two expressions follows from linear programming duality (see, e.g., Schrijver 1986).

Definition A.1. *A function $f : \mathbb{Z}^G \to \mathbb{R}$ is called* integrally convex*, if the local convex extension $\tilde{f}$ is convex.*[16] *Equivalently, f is integrally convex, if $\tilde{f} = \bar{f}$.*

The function f is called integrally concave*, if the function $-f$ is integrally convex.*

A set $S \subseteq \mathbb{Z}^G$ is an integrally convex set*, if its indicator function δ_S is integrally convex.*

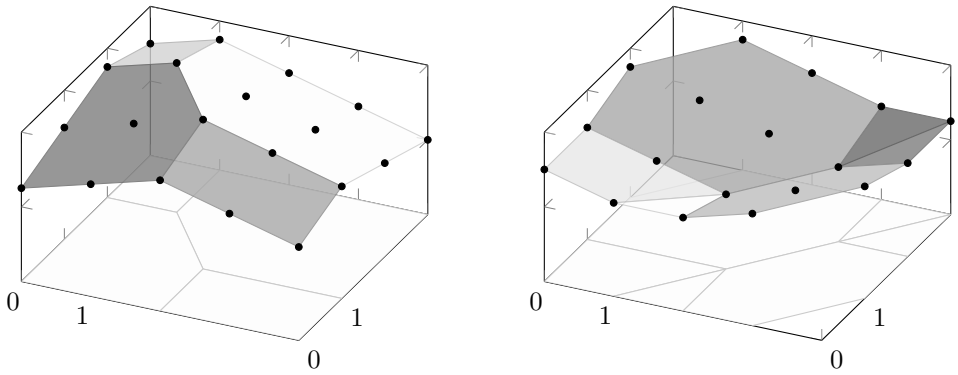
Integrally convex functions share with convex functions the important property that local minima are also global minima.

Proposition A.1. *Let $f : \mathbb{Z}^G \to \mathbb{R}$ be an integrally convex function and $x \in \mathbb{Z}^G$. Then $f(x) \leq f(y)$ for all $y \in \mathbb{Z}^G$ if and only if $f(x) \leq f(y)$ for all $y \in \mathbb{Z}^G$ with $||y - x||_\infty \leq 1$.*

Proof. We only need to show sufficiency. Consider the local convex extension $\tilde{f}$ of f. From the definition of $\tilde{f}$ in (A.5) and the local optimality of x with respect to f it follows that $\tilde{f}(x) \leq \tilde{f}(y)$ for all y with $||y - x||_\infty \leq 1$. Hence, x is a local minimum of $\tilde{f}$, which is convex because of integral convexity of f. Therefore, x is also a global minimum of $\tilde{f}$ and in particular of f. □

While integral convexity is sufficient for the global optimality of local minima, more combinatorial structure is needed for the conjugacy and duality results we need. Discrete

[16] Note that integrally convex functions are convex-extensible, but the reverse is not necessarily true. See Example 3.20 in Murota (2003).



(a) An $M^\natural$-concave function f. **(b)** An $L^\natural$-convex function g.

Figure 5: An integrally concave and convex function. Note that the sets $\arg\max_x f(x) - \langle p, x\rangle$ and $\arg\min_x g(p) - \langle p, x\rangle$ are $M^\natural$-convex and $L^\natural$-convex sets, respectively.

Convex Analysis identifies $M^\natural$-convex and $L^\natural$-convex functions as two important classes of integrally convex functions which are in one-to-one correspondence to each other under the Legendre-Fenchel transformation (defined below).

Definition A.2. *A function f is $M^\natural$-*convex*, if for $x, y \in \operatorname{dom} f$ and $j \in \operatorname{supp}^+(x-y)$*

(i) $f(x) + f(y) \geq f(x - \mathbf{1}_j) + f(y + \mathbf{1}_j)$ *or*

(ii) $f(x) + f(y) \geq f(x - \mathbf{1}_j + \mathbf{1}_k) + f(y + \mathbf{1}_j - \mathbf{1}_k)$ *for some* $k \in \operatorname{supp}^-(x-y)$.

*A function f is $M^\natural$-*concave *if the function $-f$ is $M^\natural$-convex.*

*A set $X \subseteq \mathbb{Z}^G$ is an $M^\natural$-*convex set*, if its indicator function δ_X is $M^\natural$-convex.*

The exchange property *(ii)* is closely related to the exchange axiom in matroid theory. Therefore, the M stands for "matroid."

Definition A.3. *A function g is $L^\natural$-*convex*, if for all $p, q \in \mathbb{Z}^G$ and all $\alpha \in \mathbb{Z}_+$,*

$$g(p) + g(q) \geq g([p - \alpha\mathbf{1}] \vee q) + g(p \wedge [q + \alpha\mathbf{1}]).$$

*A function g is $L^\natural$-*concave *if the function $-g$ is $L^\natural$-convex.*

*A set $P \subseteq \mathbb{Z}^G$ is an $L^\natural$-*convex set*, if its indicator function δ_P is $L^\natural$-convex.*

$L^\natural$-convex functions are precisely those submodular functions that are integrally convex. The submodularity of an $L^\natural$-convex function can be obtained by setting $\alpha = 0$ in the definition above. Since $L^\natural$-convex sets are precisely the integral sublattices of Z^G, the L stands for "lattice."

The classes of $M^\natural$- and $L^\natural$-convex functions and sets are in many ways dual to each other. First, the Legendre-Fenchel transform of an $M^\natural$-convex function is $L^\natural$-convex and vice versa. Second, the superdifferential of an $M^\natural$-concave function is an $L^\natural$-convex set, while the subdifferential of an $L^\natural$-convex function is an $M^\natural$-convex set. The superdifferential $\partial' f(x)$ of an integrally concave function $f : \mathbb{Z}^G \to \mathbb{R} \cup \{\infty\}$ at $x \in \operatorname{dom} f$ is defined as

$$\partial' f(x) = \{p \in \mathbb{Z}^G \mid f(y) - f(x) \leq \langle p, y - x\rangle \ \forall y \in \mathbb{Z}^G\}.$$

Theorem A.1. *Let f be an $M^\natural$-concave function. Then*

(i) for all $x \in \operatorname{dom} f$, the superdifferential $\partial' f(x)$ is a non-empty $L^\natural$-convex set.

(ii) for all $p \in \mathbb{Z}^G$, the set of maximizers $\arg\max_x \{f(x) - \langle p, x\rangle\}$ is an $M^\natural$-convex set if it is not empty.

An analogue version of the above theorem holds for an $M^\natural$-convex function and its subdifferential. The set of maximizers in part *(ii)* of the theorem are depicted in Figure 5a on the x_1, x_2-plane. It turns out that $M^\natural$-concave functions are characterized by either of the properties in Theorem A.1.

Define the subdifferential $\partial g(x)$ of an integrally convex function $g : \mathbb{Z}^G \to \mathbb{R} \cup \{\infty\}$ at $x \in \operatorname{dom} g$ as

$$\partial g(p) = \{x \in \mathbb{Z}^G \mid g(q) - g(p) \geq \langle x, q - p\rangle \ \forall q \in \mathbb{Z}^G\}.$$

Theorem A.2. *Let g be an $L^\natural$-convex function. Then*

(i) for all $p \in \operatorname{dom} g$, the subdifferential $\partial g(p)$ is a non-empty $M^\natural$-convex set.

(ii) for all $x \in \mathbb{Z}^G$, the set of minimizers $\arg\min_p \{g(p) - \langle p, x\rangle\}$ is an $L^\natural$-convex set if it is not empty.

An analogue version of this theorem holds for an $L^\natural$-concave function and its superdifferential. The set of $L^\natural$-convex functions is characterized by either of the properties in Theorem A.2.

The *concave conjugate or Legendre-Fenchel transform* of a function f is defined as

$$f^\circ(p) = \inf_x \{\langle p, x\rangle - f(x)\}.$$

The following conjugacy result holds for an $M^\natural$-concave function (an analogue result holds for $L^\natural$-convex functions).

Theorem A.3 (Conjugacy)**.** *Let f be an $M^\natural$-concave function. Then*

(i) the concave conjugate f° is $L^\natural$-concave and

(ii) the biconjugate of f is identical to f itself: $f^{\circ\circ} = f$.

A special property of $M^\natural$-concavity is that it is preserved under the following operation (Murota 2003). Define the *convolution* $f_1 \,\square\, f_2$ of two $M^\natural$-concave functions f_1 and f_2 through

$$(f_1 \,\square\, f_2)(x) = \max_{x^1, x^2} \ \big\{f_1(x^1) + f_2(x^2) \mid x^1 + x^2 = x\big\}. \tag{A.6}$$

We can see from the definition that the effective domain of $f_1 \,\square\, f_2$ will be the *Minkowski sum* $\operatorname{dom} f_1 + \operatorname{dom} f_2 = \{x = x^1 + x^2 \in \mathbb{R}^G \mid x^1 \in \operatorname{dom} f_1 \text{ and } x^2 \in \operatorname{dom} f_2\}$. Since the convolution is associative, we can for a collection of functions $\{f_i\}_{i \in N}$, define $f_N = \square_{i \in N} f_i$. It follows that f_N will be $M^\natural$-concave given that f_i is $M^\natural$-concave for all $i \in N$.

The Legendre-Fenchel transformation and the convolution operator satisfy the relation

$$(f_1 \,\square\, f_2)^\circ = f_1^\circ + f_2^\circ. \tag{A.7}$$

References

Arrow, K. J. and Debreu, G. (1954). Existence of equilibrium for a competitive economy, *Econometrica* **22**: 265–290.

Arrow, K. J. and Hurwicz, L. (1958). On the stability of the competitive equilibrium, I, *Econometrica* **26**: 522–552.

Ausubel, L. M. (2006). An efficient dynamic auction for heterogeneous commodities, *American Economic Review* **96**: 602–629.

Baldwin, E. and Klemperer, P. (2013). Tropical geometry to analyse demand. Working Paper, Oxford University.

Danilov, V., Koshevoy, G. and Lang, C. (2003). Gross substitution, discrete convexity and submodularity, *Discrete Applied Mathematics* **131**: 283–298.

Danilov, V., Koshevoy, G. and Murota, K. (2001). Discrete convexity and equilibria in economies with indivisible goods and money, *Mathematical Social Sciences* **41**: 251–273.

Demange, G., Gale, D. and Sotomayor, M. (1986). Multi-item auctions, *Journal of Political Economy* **94**: 863–872.

deVries, S., Schummer, J. and Vohra, R. V. (2007). On ascending vickrey auctions for heterogeneous objects, *Journal of Economic Theory* **132**(1): 95–118.

Dress, A. W. M. and Wenzel, W. (1990). Valuated matroid: A new look at the greedy algorithm, *Applied Mathematics Letters* **3**: 33–35.

Drexl, M. (2013). Substitutes and complements in trading networks. Working Paper, University of Bonn.

Fujishige, S. and Yang, Z. (2003). A note on Kelso and Crawford's gross substitutes condition, *Mathematics of Operations Research* **28**: 463–469.

Gale, D. and Shapley, L. S. (1962). College admissions and the stability of marriage, *The American Mathematical Monthly* **69**(1): 9–15.

Gul, F. and Stacchetti, E. (1999). Walrasian equilibrium with gross substitutes, *Journal of Economic Theory* **87**(1): 95–124.

Gul, F. and Stacchetti, E. (2000). The english auction with differentiated commodities, *Journal of Economic Theory* **92**(1): 66–95.

Hatfield, J. W., Kominers, S. D., Nichifor, A., Ostrovsky, M. and Westkamp, A. (2013). Stability and competitive equilibrium in trading networks, *Journal of Political Economy* . Forthcoming.

Kelso, A. S. and Crawford, V. P. (1982). Job matching, coalition formation, and gross substitutes, *Econometrica* **50**(6): 1483–1504.

Klemperer, P. (2010). The product-mix auction: A new auction design for differentiated goods, *Journal of the European Economic Association* **8**: 526–536.

McKenzie, L. (1959). On the existence of general equilibrium for a competitive market, *Econometrica* **27**: 54–71.

Milgrom, P. and Strulovici, B. (2009). Substitute goods, auctions, and equilibrium, *Journal of Economic Theory* **144**: 212–247.

Murota, K. (2003). *Discrete Convex Analysis*, SIAM Society for Industrial and Applied Mathematics, Philadelphia.

Murota, K. and Tamura, A. (2002). New characterizations of M-convex functions and their applications to economic equilibrium models with indivisibilities, *Discrete Applied Mathematics* **131**: 495–512.

Ostrovski, M. (2008). Stability in supply chain networks, *American Economic Review* **98**(3): 897–923.

Parkes, D. C. and Ungar, L. H. (2000). Iterative combinatorial auctions: Theory and practice, *Proceedings of the Seventeenth National Conference on Artificial Intelligence and Twelfth Conference on Innovative Applications of Artificial Intelligence*, pp. 74–81.

Samuelson, P. A. (1947). *Foundations of Economic Analysis*, Harvard University Press, Cambridge, MA.

Schrijver, A. (1986). *Theory of Linear and Integer Programming*, Wiley-Interscience, New York, NY.

Schrijver, A. (2000). A combinatorial algorithm minimizing submodular functions in strongly polynomial time, *Journal of Combinatorial Theory* **80**: 346–355.

Shioura, A. and Yang, Z. (2013). Equilibrium, auction, multiple substitutes and complements. Discussion Paper, University of York, York, United Kingdom.

Sun, N. and Yang, Z. (2006). Equilibria and indivisibilities: Gross substitutes and complements, *Econometrica* **74**(5): 1385–1402.

Sun, N. and Yang, Z. (2009). A double-track adjustment process for discrete markets with substitutes and complements, *Econometrica* **77**(3): 933–952.

Walras, L. (1874). *Elements of Pure Economics*, Irwin, Homewood, IL.